Superselection Sectors of $\mathfrak{so}(N)$ Wess-Zumino-Witten Models

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Abstract

The superselection structure of $\mathfrak{so}(N)$ WZW models is investigated from the point of view of algebraic quantum field theory. At level 1 it turns out that the observable algebras of the WZW theory can be constructed in terms of even CAR algebras. This fact allows to give a formulation of these models close to the DHR framework. Localized endomorphisms are constructed explicitly in terms of Bogoliubov transformations, and the WZW fusion rules are proven using the DHR sector product.

At level 2 it is shown that most of the sectors are realized in $\hat{\mathcal{H}}_{\mathrm{NS}} = \mathcal{H}_{\mathrm{NS}} \otimes \mathcal{H}_{\mathrm{NS}}$ where $\mathcal{H}_{\mathrm{NS}}$ is the Neveu-Schwarz sector of the level 1 theory. The level 2 characters are derived and $\hat{\mathcal{H}}_{\mathrm{NS}}$ is decomposed completely into tensor products of the sectors of the WZW chiral algebra and irreducible representation spaces of the coset Virasoro algebra. Crucial for this analysis is the DHR decomposition of $\hat{\mathcal{H}}_{\mathrm{NS}}$ into sectors of a gauge invariant fermion algebra since the WZW chiral algebra as well as the coset Virasoro algebra are invariant under the gauge group $O(2)$.

Zusammenfassung

Es wird die Superauswahlstruktur von $\mathfrak{so}(N)$ WZW Modellen unter dem Gesichtspunkt der algebraischen Quantenfeldtheorie untersucht. Es stellt sich heraus, daß sich für Level 1 die Observablenalgebren der WZW Theorie durch gerade CAR Algebren konstruieren lassen. Diese Tatsache erlaubt eine Formulierung für diese Modelle dicht am DHR Rahmen. Lokalisierte Endomorphismen werden explizit als Bogoliubov Transformationen konstruiert, und die WZW Fusionsregeln werden mithilfe des DHR Sektorproduktes bewiesen.

Es wird gezeigt, daß für Level 2 die meisten Sektoren realisiert sind in $\hat{\mathcal{H}}_{\mathrm{NS}} = \mathcal{H}_{\mathrm{NS}} \otimes \mathcal{H}_{\mathrm{NS}}$, wobei $\mathcal{H}_{\mathrm{NS}}$ der Neveu-Schwarz Sektor der Level 1 Theorie ist. Die Level 2 Charaktere werden abgeleitet, und $\hat{\mathcal{H}}_{\mathrm{NS}}$ wird vollständig in Tensorprodukte von Sektoren der WZW chiralen Algebra und irreduziblen Darstellungsräumen der Coset Virasoro Algebra zerlegt. Entscheidend ist bei dieser Analyse die DHR Zerlegung von $\hat{\mathcal{H}}_{\mathrm{NS}}$ in Sektoren einer eichinvarianten Fermionalgebra, da sowohl die WZW chirale Algebra als auch die Coset Virasoro Algebra unter der Eichgruppe $O(2)$ invariant sind.

Contents

Chapter 1

Introduction

This dissertation is concerned with the application of methods of algebraic quantum field theory (AQFT) to concrete models of conformal field theory (CFT). Since the basic principles and the mathematical framework of AQFT and CFT are very different it is to be feared that the number of readers being familiar with both settings is rather small. Therefore we feel obliged to give some introduction to both, AQFT and CFT. However, it is not reasonable to present largely extended reviews here; for more detailed discussions of these topics we will refer the reader to the literature.

1.1 The Algebraic Approach to Quantum Field Theory

In the early sixties Haag and Kastler [36] began to develop the algebraic approach to quantum field theory. The aim of this program was to understand quantum field theory (QFT) in a mathematically rigorous way, and this can be seen in contrast to the Lagrangian approach which is, although being very successful in high energy physics, always accompanied by serious mathematical problems. Employing only the basic principles of special relativity and quantum theory, they showed that it is natural to formulate QFT in terms of C^*-algebras which are associated to bounded regions in Minkowski spacetime and represent physical quantities that are observable by measurements within these regions. Superselection sectors, conventionally defined as (minimal) subspaces of the Hilbert space of physical states so that observables do not make transitions between them [52], arise naturally

in this framework as the inequivalent irreducible representation spaces of the algebra of observables.

1.1.1 DHR Theory: Starting from the Field Algebra

In the first paper [15] of an important series, Doplicher, Haag and Roberts (DHR for short) started their analysis from the assumption of a given field algebra and a gauge group (of the first kind) acting on it, and defined the observables to be the gauge invariant part of this field algebra. More precisely, they considered a Hilbert space $\mathcal{H}$ of physical states with associated algebra $\mathfrak{B}(\mathcal{H})$ of bounded operators. They assumed that to each bounded spacetime region $\mathcal{O}$ there is an associated von Neumann algebra $\mathfrak{F}(\mathcal{O}) \subset \mathfrak{B}(\mathcal{H})$, so that one has a net of *local field algebras* fulfilling *isotony*, i.e. $\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\mathfrak{F}(\mathcal{O}_1) \subset \mathfrak{F}(\mathcal{O}_2)$. The total field algebra $\mathfrak{F}$ is, by definition, the norm closure of the union of all local field algebras,

$$\mathfrak{F} = \overline{\bigcup_{\mathcal{O}} \mathfrak{F}(\mathcal{O})}\,,$$

and is assumed to act irreducibly on $\mathcal{H}$,

$$\mathfrak{F}'' = \mathfrak{B}(\mathcal{H})\,.$$

(A prime always denotes the commutant in the algebra of bounded operators in the corresponding Hilbert space.) Further the following assumptions were made: There is a strongly continuous unitary representation U of the covering $\tilde{\mathcal{P}}_+^\uparrow$ of the Poincaré group $\mathcal{P}_+^\uparrow$ such that the energy operator P_0 has non-negative spectrum and the eigenvalue zero belongs to a unique (up to a phase) vector $|\Omega_0\rangle \in \mathcal{H}$, the *vacuum state*. The action U of $\tilde{\mathcal{P}}_+^\uparrow$ transforms the fields covariantly,[1]

$$U(L)\,\mathfrak{F}(\mathcal{O})\,U(L)^{-1} = \mathfrak{F}(L\mathcal{O})\,, \qquad L \in \tilde{\mathcal{P}}_+^\uparrow\,,$$

and leaves the vacuum invariant, $U(L)|\Omega_0\rangle = |\Omega_0\rangle$. Further there is a strongly continuous representation Q of a compact gauge group $\mathcal{G}$, commuting with U, acting locally on $\mathfrak{F}$ in the sense that

$$Q(g)\,\mathfrak{F}(\mathcal{O})\,Q(g)^{-1} = \mathfrak{F}(\mathcal{O})\,, \qquad g \in \mathcal{G}\,,$$

[1] This assumption of a Poincaré covariant field algebra is rather restrictive, it excludes for instance Quantum Electrodynamics from the beginning. However, the setting was tailored to describe strong interaction physics with short range forces where the assumptions are coherent.

and leaving the vacuum invariant, $Q(g)|\Omega_0\rangle = |\Omega_0\rangle$. The local observable algebras $\mathfrak{A}(\mathcal{O})$ are then defined as the gauge invariant parts of $\mathfrak{F}(\mathcal{O})$,

$$\mathfrak{A}(\mathcal{O}) = \mathfrak{F}(\mathcal{O}) \cap Q(\mathcal{G})' ,$$

and the total observable algebra is

$$\mathfrak{A} = \overline{\bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})} .$$

The fields are local relatively to the observables,

$$\mathfrak{F}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{O}')' ,$$

where $\mathfrak{A}(\mathcal{O}')$ denotes the C^*-algebra generated by all $\mathfrak{A}(\mathcal{O}_1)$ with $\mathcal{O}_1$ space-like separated from $\mathcal{O}$. Note that this implies in particular locality of the observables which is the manifestation of Einstein causality of observable quantities in QFT,

$$\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{O}')' .$$

Two further assumptions were made; a certain "cluster property" and the *Reeh-Schlieder property* which states that every analytic vector for the energy operator P_0 (i.e. a vector $|\Psi\rangle \in \mathcal{H}$ such that the power series $\sum_n \|P_0^n|\Psi\rangle\| t^n/n!$ has non-zero radius of convergence in t) is cyclic and separating for each $\mathfrak{F}(\mathcal{O})$. Under these assumptions it could be shown that the Hilbert space decomposes as follows into superselection sectors $\mathcal{H}_\xi$,

$$\mathcal{H} = \bigoplus_{\xi \in \hat{\mathcal{G}}} \mathcal{H}_\xi \otimes H_\xi .$$

Here the sum runs over the spectrum $\hat{\mathcal{G}}$ of the gauge group, i.e. the set of unitary equivalence classes ξ of all (unitary, continuous) irreducible representations of $\mathcal{G}$, and each multiplicity space H_ξ carries a representation Q_ξ of class ξ with (finite) dimension d_ξ, so that $Q(g)$, $g \in \mathcal{G}$, takes the form

$$Q(g) = \bigoplus_{\xi \in \hat{\mathcal{G}}} \mathbf{1} \otimes Q_\xi(g) ,$$

whereas $A \in \mathfrak{A}$ acts as

$$A = \bigoplus_{\xi \in \hat{\mathcal{G}}} \pi_\xi(A) \otimes \mathbb{1}_{d_\xi} .$$

Here π_ξ are irreducible representations of $\mathfrak{A}$, and π_ξ and $\pi_{\xi'}$ are inequivalent for $\xi \neq \xi'$. The representation corresponding to the trivial class in $\hat{\mathcal{G}}$ is called the *vacuum representation* and is denoted by π_0. Note that the correspondence between the irreducible representations of the gauge group and those of the observable algebra induces a product of the superselection sectors of $\mathfrak{A}$ describing the composition of charge quantum numbers: It comes from the representation ring $\mathcal{R}_{\mathcal{G}} = \mathbb{Z}\hat{\mathcal{G}}$ of the gauge group and can be expressed in terms of fusion rules,

$$\xi_i \times \xi_j = \sum_k N_{ij}{}^k \, \xi_k \, ,$$

where the non-negative integers $N_{ij}{}^k$, the *fusion coefficients*, describe the multiplicity of the representation class ξ_k appearing in the decomposition of the tensor product of ξ_i and ξ_j.

With an additional maximality relation for the local observable algebras (Haag duality) and for the case that $\mathcal{G}$ is abelian it could also be shown in [15] that for all $\xi \in \hat{\mathcal{G}}$ there are certain *localized automorphisms* ϱ_ξ of $\mathfrak{A}$ which are implemented by unitaries ψ_ξ in some $\mathfrak{F}(\mathcal{O})$, $\varrho_\xi(A) = \psi_\xi A \psi_\xi^*$, $A \in \mathfrak{A}$, such that π_ξ is unitarily equivalent to π_0 composed with ϱ_ξ, $\pi_\xi \simeq \pi_0 \circ \varrho_\xi$. Later, Doplicher and Roberts [19] were able to generalize this to non-abelian sectors (i.e. sectors with $d_\xi > 1$ when the gauge group is non-abelian): These sectors correspond to non-surjective *localized endomorphisms* which are implemented by multiplets of isometries ψ_ξ^i in some $\mathfrak{F}(\mathcal{O})$, $i = 1, 2, ..., d_\xi$, transforming according to a representation of class ξ of the gauge group (and satisfying the relations of a Cuntz algebra). Since endomorphisms can be composed one gets a product of unitary equivalence classes $[\pi_\xi]$ of representations π_ξ of $\mathfrak{A}$ by defining

$$[\pi_\xi] \times [\pi_{\xi'}] = [\pi_0 \circ \varrho_\xi \varrho_{\xi'}] \, .$$

Indeed, this product precisely reproduces the above sector product, we have

$$[\pi_{\xi_i}] \times [\pi_{\xi_j}] = \bigoplus_k N_{ij}{}^k \, [\pi_{\xi_k}] \, .$$

However, first the investigations were tackled from a different direction.

1.1.2 DHR Theory: Starting from the Observables

Already in their second paper [16], Doplicher, Haag and Roberts began to investigate the theory when it is given in terms of observable algebras corresponding to the representation of $\mathfrak{A}$ in the vacuum sector. The reason for this

new direction was that it seemed to be more natural to start only from observable quantities whereas unobservable fields and the gauge group may be regarded as auxiliary objects. We will not give a complete historical review here but sketch some of the results of [16, 17, 18, 20]; for a detailed treatment of these topics we refer to Haag's book [35]. The theory is now given by a net of local von Neumann algebras $\mathcal{R}(\mathcal{O})$ acting on a Hilbert space $\mathcal{H}_0$ and labelled by $\mathcal{O} \in \mathcal{DK}$, the set of open double cones[2] in Minkowski space, and the algebra of *quasilocal observables* $\mathcal{A}$ is defined as the C^*-algebra generated by all $\mathcal{R}(\mathcal{O})$,

$$\mathcal{A} = \overline{\bigcup_{\mathcal{O}\in\mathcal{DK}} \mathcal{R}(\mathcal{O})}\,,$$

where it is assumed that $\mathcal{A}'' = \mathfrak{B}(\mathcal{H}_0)$. Since one interprets $\mathcal{R}(\mathcal{O})$ as $\pi_0(\mathfrak{A}(\mathcal{O}))$ in the previous framework, this requirement simply expresses irreducibility of the vacuum representation. One also assumes that there is a unitary, strongly continuous representation U_0 of $\mathcal{P}_+^\uparrow$ such that the observables transform covariantly and that the spectrum condition is fulfilled. The locality condition for observables will now be strengthened to *Haag duality*,

$$\mathcal{R}(\mathcal{O}) = \mathcal{A}(\mathcal{O}')'$$

where $\mathcal{A}(\mathcal{O}')$ again denotes the C^*-algebra generated by all $\mathcal{R}(\mathcal{O}_1)$ with $\mathcal{O}_1$ spacelike to $\mathcal{O}$. While being a result in the previous setting, the *DHR selection criterion*

$$\pi|_{\mathcal{A}(\mathcal{O}')} \simeq \pi_0|_{\mathcal{A}(\mathcal{O}')} \tag{1.1}$$

for some double cone $\mathcal{O}$ and $\pi_0 = id$ will now serve as a selection criterion for physical representations π that have to be considered.[3] If one denotes the unitary operator realizing the equivalence (1.1) by V then one obtains by

$$\varrho(A) = V^* A\, V\,, \qquad A \in \mathcal{A}\,,$$

[2]A double cone is a non-void intersection of a forward and a backward light-cone; the restriction to these special regions is just of a technical nature.

[3]It is now the DHR criterion which is too stringent for Quantum Electrodynamics as it excludes all states with non-vanishing charge by virtue of Gauss' law. Moreover, Buchholz and Fredenhagen pointed out [10] that also in purely massive theories there can exist charges which are measurable at arbitrarily large distances. However, such a situation can be treated [10] by replacing double cones by spacelike cones as (unbounded) localization regions, but this will not be considered here.

a localized endomorphism (i.e. $\varrho(A) = A$ whenever $A \in \mathcal{A}(\mathcal{O}'))$ satisfying

$$\pi \simeq \pi_0 \circ \varrho \,.$$

An important feature of these localized endomorphisms of the observable algebra $\mathcal{A}$ is that they allow to define a product of DHR sectors, i.e. of equivalence classes $[\pi]$ of representations π satisfying (1.1), without the presence of a gauge group a priori; namely again by $[\pi] \times [\pi'] = [\pi_0 \circ \varrho\varrho']$. So we still have a product structure of charge quantum numbers ξ which label the DHR sectors $[\pi_\xi]$. Assuming also the *Borchers property*[4] it could be shown in [17, 18] that to every charge ξ there is a unique conjugate charge $\overline{\xi}$ which means that $[\pi_0]$ appears (precisely once) in the product $[\pi_\xi] \times [\pi_{\overline{\xi}}]$. Moreover, to each charge sector there is an associated representation of the permutation group which is characterized by its statistical dimension d_ξ, the order of parastatistics, and a sign which distinguishes para-Bose from para-Fermi statistics. Thus simple sectors ($d_\xi = 1$) obey ordinary Bose or Fermi statistics. One observes that such superselection structures are completely analogous to the ring structure of (equivalence classes of) irreducible, continuous, unitary representations of compact groups: The sector product corresponds to the tensor product of group representations, and the statistical dimension to the ordinary dimension of group representations. It took some years until Doplicher and Roberts [20] succeeded in proving that the assumed structure is indeed enough to reconstruct a field net and a compact gauge group so that the observables can be recovered as the gauge invariant fields.

We have to mention that some deviations of the structure described above arise if one considers localizable charges in two-dimensional spacetime. Owing to the fact that the spacelike complement of a double cone then has two connected components one obtains, in general, braid group statistics instead of permutation symmetry of the sectors [23]. This leads to theories containing particles like "anyons" and "plektons".

1.2 Conformal Field Theory

We will give some brief introductory remarks on a few topics of CFT here, and we hope that it will make the comprehension of the following chapters

[4]The Borchers property states that for each non-zero projection E in some $\mathcal{R}(\mathcal{O})$ there is a double cone $\mathcal{O}_1$ containing $\mathcal{O}$ properly so that E is equivalent to the identity within $\mathcal{R}(\mathcal{O}_1)$.

easier. For a detailed treatment we refer the reader to the literature on CFT and its mathematical background, e.g. [30, 40] or Fuchs' book [25].

1.2.1 CFT Basics

A CFT is a quantum field theory where the fields transform covariantly under the conformal group. The conformal group is the group of transformations $f : \mathbb{M}^D \to \mathbb{M}^D$ of D-dimensional spacetime $\mathbb{M}^D$ so that the metric g transforming to $g' = g \circ f^{-1}$ remains invariant up to a scalar factor, $g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x)$, $\Omega(x) \neq 0$. The conformal group contains the Poincaré group ($\Omega(x) = 1$) as a subgroup. If $D \geq 3$ the conformal group is generated besides translations, $x'^\mu = x^\mu + a^\mu$, and Lorentz transformations, $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, also by scale transformations $x'^\mu = \lambda x^\mu$, $\lambda > 0$, and special conformal transformations

$$x'^\mu = \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2}, \qquad b^\mu \in \mathbb{R}^D .$$

These transformations generate a group of dimension $(D+1)(D+2)/2$ which, however, does not act properly on $\mathbb{M}^D$ since special conformal transformations can map finite points to infinity. A proper action can be arranged by a suitable compactification $\tilde{\mathbb{M}}^D$ of the spacetime. For $D = 2$ the situation is rather different; the conformal symmetry then is infinite dimensional. The corresponding Lie algebra consists of two copies of the Witt algebra. The Witt algebra is the Lie algebra with generators ℓ_m, $m \in \mathbb{Z}$, subject to relations

$$[\ell_m, \ell_n] = (m - n)\ell_{m+n} .$$

If the Minkowski space is compactified to $\tilde{\mathbb{M}}^2 = S^1 \times S^1$, where each circle is the image of light ray coordinates $x_\pm = x^0 \pm x^1$ via the Cayley transformation

$$x_\pm \longmapsto z, \bar{z} \in S^1, \qquad z = \frac{1 + \mathrm{i}x_+}{1 - \mathrm{i}x_+}, \qquad \bar{z} = \frac{1 + \mathrm{i}x_-}{1 - \mathrm{i}x_-},$$

then the Witt algebra action can be realized by

$$\ell_m = -z^{m+1}\frac{\mathrm{d}}{\mathrm{d}z}, \qquad \bar{\ell}_m = -\bar{z}^{m+1}\frac{\mathrm{d}}{\mathrm{d}\bar{z}}.$$

At the quantum level it is no longer the Witt algebra which implements the conformal symmetry but the Virasoro algebra $\mathfrak{Vir}$ with generators L_m, $m \in \mathbb{Z}$, and a central element C satisfying relations

$$[L_m, L_n] = (m - n)L_{m+n} + \tfrac{1}{12}\,\delta_{m,-n}\, m(m^2 - 1)\, C .$$

The Virasoro algebra is the unique non-trivial central extension of the Witt algebra. The extension is necessary because one requires the conformal energy operator $H = L_0 + \bar{L}_0$ to be bounded from below. Here $\bar{L}_0 \in \overline{\mathfrak{Vir}}$, the second copy of the Virasoro algebra. In general, the Hilbert space of physical states of a two-dimensional CFT splits as

$$\mathcal{H}_{\text{phys}} = \bigoplus_{i,j} \mathcal{H}_i \otimes \bar{\mathcal{H}}_j$$

where $\mathcal{H}_i$ and $\bar{\mathcal{H}}_j$ carry unitary irreducible representations of $\mathfrak{Vir}$ and $\overline{\mathfrak{Vir}}$. Let us consider one copy of the Virasoro algebra alone for a while. Unitarity means that $L_m^* = L_{-m}$. The positive energy condition now means that the spaces $\mathcal{H} \equiv \mathcal{H}_i$ have to be *highest weight modules*, i.e. there is a vector $|\Omega_\Delta\rangle \in \mathcal{H}$ such that

$$L_0|\Omega_\Delta\rangle = \Delta|\Omega_\Delta\rangle$$

with a scalar Δ, and

$$L_m|\Omega_\Delta\rangle = 0\,, \qquad m > 0\,.$$

By irreducibility, the central element C acts as a scalar c. Moreover, $\mathcal{H}$ is spanned by vectors

$$L_{-m_1}L_{-m_2}\cdots L_{-m_k}|\Omega_\Delta\rangle\,, \qquad m_1 \geq m_2 \geq \cdots \geq m_k > 0\,. \tag{1.2}$$

In general these vectors span the *Verma module* $M(c, \Delta)$ freely. However, the unitarity condition may imply linear dependencies of vectors (1.2), or, equivalently, the existence of *null states* with vanishing norm. Unitarity imposes also restrictions on the possible values of c and Δ.

Let us now turn to the picture of quantum fields acting in the Hilbert space $\mathcal{H}_{\text{phys}}$. We first would like to emphasize that unitary modules over Lie algebras possess a pre-Hilbert space structure and are, for example, generated by the finite linear span of vectors like (1.2). Since in quantum physics one has to deal with Hilbert spaces we will use the corresponding Hilbert space completions so that the modules appear as their dense subspaces. When there is no confusion with domains of unbounded operators etc. we will sometimes sloppily use the term “modules” also for the completed spaces. The first field one may introduce is the conformal *stress energy tensor*, defined by the (formal) series

$$T(z) = \sum_{m\in\mathbb{Z}} z^{-m-2}L_m$$

and analogously $\bar{T}(\bar{z})$. Very important objects in CFT are the *primary fields*. A primary field $\phi(z,\bar{z})$ of weight $(\Delta,\bar{\Delta})$ transforms, by definition, covariantly relative to the stress energy tensor, i.e.

$$\begin{aligned} [L_m, \phi(z,\bar{z})] &= z^m \left(z\frac{\partial}{\partial z} + (m+1)\Delta \right) \phi(z,\bar{z})\,, \\ [\bar{L}_m, \phi(z,\bar{z})] &= \bar{z}^m \left(\bar{z}\frac{\partial}{\partial \bar{z}} + (m+1)\bar{\Delta} \right) \phi(z,\bar{z})\,. \end{aligned} \tag{1.3}$$

Note that the stress energy tensor itself is not a primary field. The fields are understood to be "analytically continued" to $\mathbb{C} \setminus \{0\}$ in both variables $z, \bar{z}$. In a fixed CFT the value of c is fixed and it is also assumed that there is a vacuum vector $|\Omega_0\rangle \in \mathcal{H}_{\mathrm{phys}}$ of highest weight zero, $L_0|\Omega_0\rangle = \bar{L}_0|\Omega_0\rangle = 0$. An important feature of a primary field ϕ with weight $(\Delta,\bar{\Delta})$ is that it creates a highest weight vector with weight $(\Delta,\bar{\Delta})$ out of the vacuum in the sense that

$$\lim_{z,\bar{z}\to 0} \phi(z,\bar{z})|\Omega_0\rangle = |\Omega_{\Delta,\bar{\Delta}}\rangle\,.$$

Thus we have a state-field-correspondence between highest weight states and primary fields. Similarly, fields that correspond to non-highest weight vectors are called *descendants*. For the primary fields one introduces operator product expansions,

$$\phi_i(z,\bar{z})\phi_j(w,\bar{w}) = \sum_k {c_{ij}}^k(z,\bar{z},w,\bar{w})\phi_k(w,\bar{w}) + \ldots$$

in terms of radially ordered products, i.e. $|z| > |w|$, $|\bar{z}| > |\bar{w}|$. Here the dots stand for contributions of descendants. Conformal covariance implies that

$${c_{ij}}^k(z,\bar{z},w,\bar{w}) = (z-w)^{\Delta_k - \Delta_i - \Delta_j} (\bar{z}-\bar{w})^{\bar{\Delta}_k - \bar{\Delta}_i - \bar{\Delta}_j}\, {C_{ij}}^k\,.$$

The constants ${C_{ij}}^k$ may be interpreted as an analogy to Clebsch-Gordan coefficients. Somewhat coarser information is encoded in the *fusion rules*

$$\phi_i \times \phi_j = \sum_k {N_{ij}}^k \phi_k$$

where the (non-negative integer) fusion coefficients indicate whether the primary field ϕ_k appears in the operator product expansion of ϕ_i and ϕ_j (${N_{ij}}^k \neq 0$) or not (${N_{ij}}^k = 0$). The primary fields of a CFT model generate an associated *fusion ring* $\mathcal{R}$ (over $\mathbb{Z}$), see e.g. [26]. Since such fusion rings

are abelian there are only one-dimensional representations. There is a distinguished positive representation $\mathcal{D}$ which associates to each primary field ϕ_i a *quantum dimension* $\mathcal{D}_i \equiv \mathcal{D}(\phi_i) > 0$.

In generic CFTs the two copies $\mathfrak{Vir} \times \overline{\mathfrak{Vir}}$ are extended to a larger *symmetry algebra* $\mathfrak{W} \times \overline{\mathfrak{W}}$ and each *chiral half* $\mathfrak{W}, \overline{\mathfrak{W}}$ contains a Virasoro algebra as a subalgebra. In such theories there can appear fusion coefficients $N_{ij}{}^k > 1$. This corresponds to the fact that there might be additional contributions of descendants in the operator product expansions that would be excluded in the case $N_{ij}{}^k = 1$ by $\mathfrak{W}, \overline{\mathfrak{W}}$-symmetry, see e.g. [25, Sect. 5.1]. Examples of CFTs with enlarged symmetry algebras are the *Wess-Zumino-Witten* (WZW) theories that will be discussed now. Moreover, one may treat each chiral algebra for its own so that one has a chiral CFT on the circle.

1.2.2 WZW Theories

In a WZW theory the chiral algebra $\mathfrak{W}$ is given by the semi-direct sum of an (untwisted) affine Kac-Moody algebra $\hat{\mathfrak{g}}$ and an associated Virasoro algebra. This has to be made a little bit more precise now. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra. Recall that $\mathfrak{g}$ has a canonical *triangular decomposition*,

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$$

where the commutative *Cartan subalgebra* $\mathfrak{h}$ has a basis $\{H^j,\ j = 1, 2, \dots, \ell\}$ and the subalgebras $\mathfrak{g}_\pm$ are generated by Chevalley generators $E^j_\pm$, $j = 1, 2, \dots, \ell$, subject to commutation relations

$$[H^j, H^k] = 0\,, \quad [E^j_+, E^k_-] = \delta_{j,k} H^j\,, \quad [H^j, E^k_\pm] = \pm(\alpha^{(k)})^j E^k_\pm\,,$$

where the $\alpha^{(k)}$ are the simple roots of the Lie algebra $\mathfrak{g}$. The associated (infinite-dimensional) affine Lie algebra $\hat{\mathfrak{g}}$ is generated by a central element K and the range of linear mappings $J_m : \mathfrak{g} \to \hat{\mathfrak{g}}$, $T \mapsto J_m(T)$, $m \in \mathbb{Z}$, so that commutation relations $[J_m(T), K] = 0$ and

$$[J_m(T), J_n(T')] = J_{m+n}([T, T']) + m\, \delta_{m,-n} (T|T') K$$

hold. Here $(\cdot|\cdot)$ denotes the invariant bilinear form of $\mathfrak{g}$. Adding one further element D, the "derivation", satisfying

$$[D, J_m(T)] = m\, J_m(T)\,, \qquad [D, K] = 0$$

one obtains the full affine Kac-Moody algebra according to Cartan's classification, for simplicity, we will also refer to $\hat{\mathfrak{g}}$ as the affine Lie algebra when the derivation is omitted, however. The affine Lie algebra inherits also a triangular decomposition,

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{g}}_-$$

where $\hat{\mathfrak{g}}_\pm = J_0(\mathfrak{g}_\pm) \oplus \bigoplus_{m=1}^{\infty} J_{\pm m}(\mathfrak{g})$ and $\hat{\mathfrak{h}} = J_0(\mathfrak{h}) \oplus \mathbb{C}K \oplus \mathbb{C}D$. Similarly, one has Chevalley generators $\mathcal{E}^j_\pm \in \hat{\mathfrak{g}}_\pm$, $j = 0, 1, \dots, \ell$,

$$\mathcal{E}^j_\pm = J_0(E^j_\pm), \quad j = 1, 2, \dots, \ell\,, \qquad \mathcal{E}^0_\pm = \pm J_{\pm 1}(E_{\pm\theta})\,.$$

Here $E_\theta \in \mathfrak{g}$ is the element corresponding to the highest root θ of $\mathfrak{g}$. We also introduce $\mathcal{H}^j = J_0(H^j)$, $j = 1, 2, \dots, \ell$. It is the triangular decomposition which allows to define highest weight modules over an affine Kac-Moody algebra quite parallel to those over simple Lie algebras. A highest weight module over an affine Lie algebra $\hat{\mathfrak{g}}$ is a vector space $\mathcal{H}_\Lambda$ with a highest weight vector $|\Omega_\Lambda\rangle \in \mathcal{H}_\Lambda$ that is annihilated by the Chevalley generators of positive grade,

$$\mathcal{E}^j_+|\Omega_\Lambda\rangle = 0\,, \qquad j = 0, 1, \dots, \ell\,,$$

the Cartan subalgebra acts by scalars on $|\Omega_\Lambda\rangle$,

$$\mathcal{H}^j|\Omega_\Lambda\rangle = \Lambda^j|\Omega_\Lambda\rangle\,, \qquad K|\Omega_\Lambda\rangle = k|\Omega_\Lambda\rangle\,,$$

and the action of $\hat{\mathfrak{g}}_-$ on $|\Omega_\Lambda\rangle$ spans the whole vector space $\mathcal{H}_\Lambda$. For *integrable* highest weight modules (i.e. those modules which admit a certain definition of an exponential map) the normalized eigenvalue of K, called the *level*,

$$k^\vee = \frac{2k}{\theta^2}\,,$$

(θ^2 is the square length of the highest root of $\mathfrak{g}$) can only be a non-negative integer. The interesting highest weight modules for CFT are the unitary ones, i.e. those which possess a pre-Hilbert space structure such that $(\mathcal{E}^j_+)^* = \mathcal{E}^j_-$ and $(\mathcal{H}^j)^* = \mathcal{H}^j$. From the unitarity requirement arise severe restrictions on the possible weights $\Lambda = (\Lambda^j)$: At a fixed level $k^\vee$, there is only a finite number of admissible weights Λ.

It is the *Sugawara construction* which associates to an affine Lie algebra a realization of the Virasoro algebra. This works as follows: Fix a basis

$\{T^a,\, a = 1, 2, \dots, \dim \mathfrak{g}\}$ of $\mathfrak{g}$ such that $(T^a|T^b) = \delta_{a,b}$ and set $J^a_m = J_m(T^a)$. In particular in a highest weight module the following expression is well defined,

$$L_m = \frac{1}{\theta^2(k^\vee + g^\vee)} \sum_{a=1}^{\dim \mathfrak{g}} \sum_{n\in\mathbb{Z}} :J^a_n J^a_{m-n}: \,, \qquad m \in \mathbb{Z}\,.$$

Here $g^\vee$ is the dual Coxeter number of $\mathfrak{g}$, and we used the normal ordering prescription

$$:J^a_m J^a_n: \; = \left\{ \begin{array}{ll} J^a_m J^a_n & \quad m < 0 \\ J^a_n J^a_m & \quad m \geq 0 \end{array} \right. .$$

One checks that

$$[L_m, J^a_n] = -n\, J^a_{m+n}\,, \tag{1.4}$$

in particular, the generator $-L_0$ can be identified with the derivation D. Moreover, at a fixed level $k^\vee$, the L_m satisfy the relations of the Virasoro algebra with fixed value of the central charge

$$c = \frac{k^\vee \dim \mathfrak{g}}{k^\vee + g^\vee}\,.$$

As mentioned, for a WZW theory the chiral algebra is given by

$$\mathfrak{W} = \hat{\mathfrak{g}} \rtimes \mathfrak{Vir}$$

acting on a (pre-) Hilbert space which is the direct sum over the unitary highest weight modules of $\hat{\mathfrak{g}}$ at a fixed level, and $\mathfrak{Vir}$ is given by the Sugawara construction. (If we consider an affine Lie algebra $\hat{\mathfrak{g}}$ concretely realized at a fixed level $k^\vee$ it will often be denoted by $\hat{\mathfrak{g}}_{k^\vee}$.) Note that the *currents*

$$J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} J^a_n$$

from (1.4) in comparison with (1.3) are (chiral) primary fields of unit weight. Physicists often use the term "current algebras" instead of affine Lie algebras.

Also very important objects in CFT are the *Virasoro specialized characters*. By positivity of L_0 one can define in a module $\mathcal{H}_i$ of the chiral algebra

$$\chi_i(\tau) = \mathrm{tr}_{\,\mathcal{H}_i} q^{L_0 - \frac{c}{24}} \tag{1.5}$$

where $q = \exp(2\pi \mathrm{i}\tau)$ and τ is in the upper complex half plane. Such characters possess simple transformation properties with respect to the *modular group* $PSL(2;\mathbb{Z})$ which can be generated by elements S, T with relations $S^2 = (ST)^3 = 1$ and realized by transformations $T : \tau \mapsto \tau + 1$ and $S : \tau \mapsto -1/\tau$. The characters form a (projective) representation of $PSL(2;\mathbb{Z})$ in the sense that S acts as a matrix (S_{ij}),

$$\chi_i\left(-\frac{1}{\tau}\right) = \sum_j S_{ij}\,\chi_j(\tau)$$

whereas T acts diagonally,

$$\chi_i(\tau+1) = \exp\left(2\pi\mathrm{i}\left(\Delta_i - \tfrac{c}{24}\right)\right)\chi_i(\tau)\,.$$

Such characters are very helpful tools in the analysis of CFTs. There is an amazing connection between the modular transformation matrix S and the fusion rules; namely S "diagonalizes the fusion rules" in the sense that

$$N_{ij}{}^{k} = \sum_l \frac{S_{il}S_{jl}(S^{-1})_{lk}}{S_{0l}}\,.$$

This is called the Verlinde formula (since it goes back to a conjecture of E. Verlinde), and sometimes it is convenient to use this formula for the explicit computation of the fusion rules of WZW models.

1.2.3 Coset Theories

An important tool to generate new CFTs from WZW theories is the *coset construction* of Goddard, Kent and Olive [31, 32]. We will briefly sketch some of the ideas of [31, 32] and also [38] here. First we have to note that the Sugawara construction can be generalized to affine Lie algebras $\hat{\mathfrak{g}}$ based on semisimple Lie algebras $\mathfrak{g}$, i.e. Lie algebras of the form $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_n$ with $\mathfrak{g}_i$ simple. Modules of such $\hat{\mathfrak{g}}$ are typically of the form $\mathcal{H} = \bigotimes_{i=1}^n \mathcal{H}_i$ where $\mathcal{H}_i$ are modules of the affine Lie algebras $\hat{\mathfrak{g}}_i$ associated to $\mathfrak{g}_i$. Suppose that the $\mathcal{H}_i$ are (unitary) highest weight modules of $\hat{\mathfrak{g}}_i$ at fixed levels k_i. Then we can define

$$L_m^{\mathfrak{g}} = \sum_{i=1}^n L_m^{\mathfrak{g}_i}$$

where $L_m^{\mathfrak{g}_i}$ is the Virasoro operator associated to $\hat{\mathfrak{g}}_i$ (acting non-trivially on the i-th tensor factor $\mathcal{H}_i$). Such operators satisfy the relations of a Virasoro algebra with central charge $c^{\mathfrak{g}} = \sum_{i=1}^n c^{\mathfrak{g}_i}$. Now suppose that there is a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ with associated affine Lie algebra $\hat{\mathfrak{h}}$ and Virasoro operators $L_m^{\mathfrak{h}}$. The operators

$$L_m^{\mathfrak{g}/\mathfrak{h}} = L_m^{\mathfrak{g}} - L_m^{\mathfrak{h}}$$

generate the *coset Virasoro algebra* $\mathfrak{Vir}^{\mathfrak{g}/\mathfrak{h}}$ which is a Virasoro algebra of central charge $c^{\mathfrak{g}/\mathfrak{h}} = c^{\mathfrak{g}} - c^{\mathfrak{h}}$. Further one checks

$$[L_m^{\mathfrak{g}/\mathfrak{h}}, J_n^a] = 0 \tag{1.6}$$

where J_n^a, $a = 1, 2, \dots, \dim \mathfrak{h}$, are the generators of $\hat{\mathfrak{h}}$. Let $\mathcal{H}_\Lambda$ be a highest weight module over $\hat{\mathfrak{g}}$ of weight Λ. As a module over $\hat{\mathfrak{h}}$, one can decompose $\mathcal{H}_\Lambda$ as

$$\mathcal{H}_\Lambda = \bigoplus_\lambda \mathcal{H}_{\Lambda;\lambda} \otimes \mathcal{H}_\lambda^{\hat{\mathfrak{h}}} \tag{1.7}$$

into tensor products of branching spaces $\mathcal{H}_{\Lambda;\lambda}$ and highest weight modules $\mathcal{H}_\lambda^{\hat{\mathfrak{h}}}$ over $\hat{\mathfrak{h}}$. The branching spaces $\mathcal{H}_{\Lambda;\lambda}$ constitute modules of $\mathfrak{Vir}^{\mathfrak{g}/\mathfrak{h}}$ due to (1.6); they are in general not irreducible but can be interpreted as the representation spaces of an enlarged algebra $\mathfrak{Cos}$ of a certain coset CFT. On the level of characters, the decomposition (1.7) reads

$$\chi_\Lambda(\tau) = \sum_\lambda b_{\Lambda;\lambda}(\tau)\, \chi_\lambda^{\hat{\mathfrak{h}}}(\tau)\,,$$

and the *branching functions* $b_{\Lambda;\lambda}$ are to be interpreted as (sums of) the characters of the coset CFT.

Let us illustrate this at a special example. Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{n}$ with $\mathfrak{n}$ a simple Lie algebra. Consider a (highest weight) module over $\hat{\mathfrak{g}}$ of the form

$$\mathcal{H}_{\Lambda,\Lambda'} = \mathcal{H}_\Lambda^{(1)} \otimes \mathcal{H}_{\Lambda'}^{(k)}$$

where $\mathcal{H}_\Lambda^{(1)}$ is a level 1, and $\mathcal{H}_{\Lambda'}^{(k)}$ is a level k module over $\hat{\mathfrak{n}}$. Further, let $\mathfrak{h}$ be the diagonal embedding of $\mathfrak{n}$ in $\mathfrak{g}$. Then $\hat{\mathfrak{h}}$ acts on $\mathcal{H}_{\Lambda,\Lambda'}$ at level $k+1$. According to (1.7) we may decompose $\mathcal{H}_{\Lambda,\Lambda'}$ into highest weight modules $\mathcal{H}_\lambda^{(k+1)}$ over $\hat{\mathfrak{h}}_{k+1}$,

$$\mathcal{H}_{\Lambda,\Lambda'} = \bigoplus_\lambda \mathcal{H}_{\Lambda,\Lambda';\lambda} \otimes \mathcal{H}_\lambda^{(k+1)}$$

where the branching spaces $\mathcal{H}_{\Lambda,\Lambda';\lambda}$ constitute modules of $\mathfrak{Cos}$ of the coset theory which is in this special case denoted by $(\hat{\mathfrak{n}}_1 \oplus \hat{\mathfrak{n}}_k)/\hat{\mathfrak{n}}_{k+1}$. For the characters the branching reads

$$\chi_{\Lambda,\Lambda'}(\tau) \equiv \chi^{(1)}_{\Lambda}(\tau)\, \chi^{(k)}_{\Lambda'}(\tau) = \sum_{\lambda} b_{\Lambda,\Lambda';\lambda}(\tau)\, \chi^{(k+1)}_{\lambda}(\tau)\,.$$

Here $\chi^{(1)}_{\Lambda}$, $\chi^{(k)}_{\Lambda'}$, $\chi^{(k+1)}_{\lambda}$ denote the characters of highest weight modules over $\hat{\mathfrak{n}}$ at levels 1, k, $k+1$, respectively, and the branching functions $b_{\Lambda,\Lambda';\lambda}$ are characters of the coset CFT $(\hat{\mathfrak{n}}_1 \oplus \hat{\mathfrak{n}}_k)/\hat{\mathfrak{n}}_{k+1}$. In [32] this procedure was successfully used to realize the full discrete series of highest weight modules over $\mathfrak{Vir}$ with central charge $0 < c < 1$ with $\mathfrak{n} = \mathfrak{su}(2)$. In Chapter 4 we will concentrate to the special coset CFT $(\widehat{\mathfrak{so}}(N)_1 \oplus \widehat{\mathfrak{so}}(N)_1)/\widehat{\mathfrak{so}}(N)_2$.

1.3 CFT within the Algebraic Approach

It is perhaps not too surprising that there are always difficulties to apply the abstract and conceptually clear mathematical setting of AQFT to concrete quantum field theoretical models. However, models of CFT seem to be a fruitful area of application of the algebraic methods. Although the technical tools that are used in the algebraic approach and in CFT are rather different, a lot of structural similarities has been observed for a long time. The appearance of braid group statistics in two-dimensional spacetime is only one aspect. Indeed, the symmetry (or chiral) algebras seem to play a rôle parallel to that of the observable algebras in the algebraic approach, and their highest weight modules appear as the perfect analogue to the superselection sectors. Therefore a suitable formulation of CFT models in the algebraic framework is expected to reproduce the conformal fusion rule structure by the DHR sector product. Unfortunately, these natural translation prescriptions do not have the status of a proven mathematical theorem. Moreover, certain deviations of the canonical DHR framework will necessarily arise because the fusion rings of CFT models are in general not associated to DHR gauge groups but to more general "quantum symmetry" objects like quantum groups. For instance, the quantum dimensions $\mathcal{D}$ which would correspond to the representation dimension (or statistical dimension) in the DHR framework are in general non-integral. But there is a small number of CFT models which have been successfully investigated in the algebraic framework and where these

correspondences could be established. These models are chiral theories, so we have to discuss how the algebraic framework looks on the circle.

1.3.1 DHR Theory on the Circle

We have seen that the circle S^1 as a "spacetime" arises as the compactification of a light cone axis. Since double cones in $\mathbb{M}^2$ project to intervals on the light cone, the natural localization regions on the circle are non-void, open proper subintervals $I \subset S^1$; the set of those will be denoted by $\mathcal{J}$. We give a formulation starting from the observables in the vacuum sector parallel to the discussion in Subsection 1.1.2. Acting in some Hilbert space $\mathcal{H}_0$ we have for each $I \in \mathcal{J}$ a local von Neumann algebra $\mathcal{R}(I)$ so that isotony holds, $I_1 \subset I_0$ implies $\mathcal{R}(I_1) \subset \mathcal{R}(I_0)$. Instead of the Poincaré group here the group $PSU(1,1)$ acts as the spacetime symmetry,[5] i.e. there is a strongly continuous representation U of $PSU(1,1)$ in $\mathcal{H}_0$. The observables transform covariantly,

$$U(g)\mathcal{R}(I)U(g)^{-1} = \mathcal{R}(gI)\,, \qquad g \in PSU(1,1)\,,$$

and the generator L_0 of rotations has non-negative spectrum. The eigenvalue zero belongs to a unique vacuum vector $|\Omega_0\rangle \in \mathcal{H}_0$. Since spacelike separated double cones project to disjoint intervals, the locality requirement becomes now

$$\mathcal{R}(I_1) \subset \mathcal{R}(I_2)'\,, \qquad I_1 \cap I_2 = \emptyset\,.$$

A total algebra cannot be defined as the norm closure of the union of all local algebras since the set $\mathcal{J}$ is not directed by inclusion. However, for $\zeta \in S^1$, the set $\mathcal{J}_\zeta$ of intervals $I \in \mathcal{J}$ that do not contain ζ in their closure is directed; we obtain a bundle of C^*-algebras

$$\mathcal{A}_\zeta = \overline{\bigcup_{I \in \mathcal{J}_\zeta} \mathcal{R}(I)}\,,$$

each invariant with respect to the subgroup of $PSU(1,1)$ leaving the point ζ invariant. Following [24], one could also introduce the universal algebra; however, for our applications it turned out to be more convenient to restrict to a fixed algebra $\mathcal{A}_\zeta$. We call such an algebra $\mathcal{A}_\zeta$ the observable algebra of the punctured circle, the point ζ will then be called "point at infinity".

[5] The explicit action of $PSU(1,1)$ on the circle will be discussed in Chapter 3, Subsection 3.1.4.

1.3.2 Models

The first example of a CFT treated in the algebraic framework was the $\mathfrak{u}(1)$ current algebra on the circle [11], generated by a current $J(z)$, $z \in S^1$, with commutation relations

$$[J(z), J(z')] = -\delta'(z - z')\,,$$

or, in the language of Fourier-Laurent components

$$[J_m, J_n] = m\,\delta_{m,-n}\,, \qquad J(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} J_n\,.$$

The sectors of the theory are the highest weight modules of the $\mathfrak{u}(1)$ current algebra and are specified by a highest weight $\lambda \in \mathbb{R}$ which we call the charge in this context. Introducing *Weyl operators*

$$W(f) = \mathrm{e}^{\mathrm{i}J(f)}\,, \qquad J(f) = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}}\, J(z) f(z)\,,$$

for real test functions f on the circle, one arrives at a C^*-algebra description and has relations

$$W(f)W(g) = \mathrm{e}^{-A(f,g)/2} W(f+g)\,, \qquad A(f,g) = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}}\, f'(z) g(z)\,.$$

Local observable algebras are then defined to be generated by Weyl operators $W(f)$ with locally supported test functions f. The relevant endomorphisms ϱ_q (which are indeed automorphisms here) are induced by certain "charge functions" q on S^1 with $zq(z) \in \mathbb{R}$; they are defined by their action on Weyl operators

$$\varrho_q(W(f)) = \mathrm{e}^{\mathrm{i}q(f)} W(f)\,, \qquad q(f) = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}}\, q(z) f(z)\,.$$

Such endomorphisms indeed generate a sector of charge

$$\lambda_q = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}}\, q(z)\,.$$

A somewhat more interesting model, namely the chiral Ising model, was first investigated in [41] from the algebraic point of view. This model is,

roughly speaking, given by a $c = 1/2$ Virasoro algebra, and its three inequivalent, unitary, irreducible highest weight modules are realized in fermionic representation spaces. More precisely, there appear two different representations of the canonical anticommutation relations (CAR), the *Neveu-Schwarz* and the *Ramond sector*. The $\mathfrak{Vir}$-modules arise as their subspaces related to the fact that the Virasoro generators act as unbounded expressions in fermionic creation and annihilation operators. This explicit realization allowed to employ the structure of the underlying fermions to construct the relevant endomorphisms which indeed reproduced the Ising fusion rules. However, the use of observable algebras containing unbounded elements and also the use of non-localized endomorphisms did not really fit into the DHR framework so that some open questions were left. In particular, non-localized endomorphisms do not generate a unique fusion ring. A formulation more close to the DHR framework was given in [5]. The foundation of the analysis carried out there is the fact that the representation theory of the even CAR algebra (i.e. the subalgebra of the CAR algebra generated by bilinear expressions in creation and annihilation operators) reproduces precisely the modules of the $c = 1/2$ Virasoro algebra. Therefore local even CAR algebras could be identified as local observable algebras and the localized endomorphisms that generate the sectors could be defined in terms of Bogoliubov transformations of the CAR algebra, moreover, since local normality could be proven, their action could be lifted to a net of local von Neumann algebras. Having the well-developed mathematics of Bogoliubov transformations and quasi-free states of CAR at hand, various equivalences could be proven; as a result, the Ising fusion rule algebra was rigorously verified. We provide a generalization of this analysis to the level 1 $\mathfrak{so}(N)$ WZW models in this dissertation in Chapter 3.

There is another important operator algebraic approach to CFT, namely Wassermann's analysis [51] of fusion of positive energy representations using bounded operators. Although making essential use of the fermionic construction in WZW models, the algebras of bounded operators do not arise as certain fermion algebras in this analysis but from representations of the Lie group which corresponds to the chiral (Lie) algebra. The Lie group associated to a WZW model with chiral algebra $\hat{\mathfrak{g}} \rtimes \mathfrak{Vir}$ is given by a semidirect product $\mathcal{L}G \rtimes \mathrm{Diff}S^1$ where $\mathcal{L}G$ is a central extension of the loop group LG and $\mathrm{Diff}S^1$ is the diffeomorphism group of the circle. Let G be a compact Lie group with associated Lie algebra $\mathfrak{g}$. The elements of the loop group LG are the smooth mappings from the circle S^1 into the group G ("loops"),

and the group product is given by point-wise multiplication [46]. The interesting representations of LG are those of *positive energy* i.e. unitary, projective representations that extend to $LG \rtimes \mathrm{Diff}S^1$ so that the generator of rotations is bounded from below. Such projective representations come from genuine representations of $\mathcal{L}G$, a central extension of LG by the circle group $\mathbb{T}$. Indeed, the positive energy representations of LG are just the "integrated versions" of the unitary integrable highest weight modules of the Lie algebra $\hat{\mathfrak{g}} \rtimes \mathfrak{Vir}$. Wassermann's analysis essentially concentrates on the special case $G = SU(N)$ where the positive energy representations of the loop group $LSU(N)$ are realized in fermionic Fock spaces. Local algebras arise in the framework of loop groups naturally as the von Neumann algebras which are generated by *local loop groups* L_IG in a certain representation; for some interval $I \subset S^1$ the local loop group L_IG consists of those loops equal to the identity off I. However, Wassermann does not use endomorphisms to fuse the positive energy representations of $LSU(N)$ but the so-called "Connes fusion" coming from the theory of bimodules over von Neumann algebras. With this machinery he can indeed reproduce the fusion rules of $LSU(N)$ at arbitrary level given by the Verlinde formula. Moreover, he can prove that the Connes fusion is equivalent to the DHR sector product of localized endomorphisms. An approach to the discrete series representations of $\mathrm{Diff}S^1$ following Wassermann's ideas is given in [39].

1.4 Summary of Results and Overview

We have already discussed that the unitary highest weight modules of the chiral algebra play the rôle of the superselection sectors of the WZW theory. In this dissertation we investigate the superselection structure of $\mathfrak{so}(N)$ WZW models at level 1 and 2. At level 1 we are able to present an operator algebraic formulation of these models close to the DHR framework. We did not succeed in developing an analogous program at level 2, however, we will show that the mathematical methods of the DHR analysis can be successfully applied to figure out a lot of information on the superselection structure of the WZW models at level 2 and, possibly, also at higher level.

1.4.1 Results (Level 1)

At level 1 the $\mathfrak{so}(N)$ WZW models have only a small number of sectors, the basic ($\circ$), the vector (v) and two spinor modules (s, c) if N is even, $N = 2\ell$, respectively the basic, the vector and one spinor module (σ) if N is odd, $N = 2\ell + 1$. It is known that these modules are realized in two representation spaces of CAR, namely in the Neveu-Schwarz sector $\mathcal{H}_{\mathrm{NS}}$ ($\circ$, v) and in the Ramond sector $\mathcal{H}_{\mathrm{R}}$ (s, c resp. σ). For the explicit realization, we use Araki's selfdual formalism of CAR. The selfdual CAR algebra $\mathcal{C}(\mathcal{K},\Gamma)$ is the C^*-algebra generated by elements $B(f)$, B linear in vectors f of some Hilbert space $\mathcal{K}$ which is endowed with a complex conjugation Γ, so that $\{B(f)^*, B(g)\} = \langle f, g\rangle \mathbf{1}$ and $B(f)^* = B(\Gamma f)$, $f, g \in \mathcal{K}$, holds. The notion of creation and annihilation operators appears in this framework only in a chosen (Fock) representation of $\mathcal{C}(\mathcal{K},\Gamma)$. In our case the underlying "pre-quantized" Hilbert space is $\mathcal{K} = L^2(S^1;\mathbb{C}^N)$. In $\mathcal{H}_{\mathrm{NS}}$ and $\mathcal{H}_{\mathrm{R}}$ act representations π_{NS} respectively π_{R} of $\mathcal{C}(\mathcal{K},\Gamma)$, and the actions of $\widehat{\mathfrak{so}}(N)$ in $\mathcal{H}_{\mathrm{NS}}$ and $\mathcal{H}_{\mathrm{R}}$ arise via second quantization from a natural action in the pre-quantized Hilbert space. The second quantized currents and also their Sugawara operators possess expressions as sums over bilinears in creation and annihilation operators. Using some well-known facts on the CAR algebra and applying also some of our own results on the decomposition of certain CAR representations into irreducibles, we can show that the representations of the even CAR algebra $\mathcal{C}(\mathcal{K},\Gamma)^+$ (the subalgebra generated by bilinear expressions $B(f)B(g)$, $f, g \in \mathcal{K}$) reproduce precisely the sectors of the chiral algebra in $\mathcal{H}_{\mathrm{NS}}$ and $\mathcal{H}_{\mathrm{R}}$. Thus we can identify the even CAR algebra as observables, and a local net structure comes from the natural embedding of the underlying pre-quantized spaces $L^2(I;\mathbb{C}^N)$ with $I \subset S^1$ open intervals. Thereby we distinguish a point at infinity $\zeta \in S^1$ and restrict to "finite intervals" i.e. those intervals such that ζ is not contained in their closures. The WZW sectors then appear as irreducible summands of the restrictions of π_{NS} and π_{R} to $\mathcal{C}(\mathcal{K},\Gamma)^+$; the vacuum sector of the theory corresponds to the basic module. Therefore the localized endomorphisms that generate the relevant sectors can be defined as endomorphisms of the (even) CAR algebra; we can realize them explicitly in terms of Bogoliubov transformations. Using again CAR mathematics, we can prove that their composition realizes the WZW fusion rules. A crucial point is that *local normality* holds, i.e. we prove that the representations which represent the sectors are quasi-equivalent in restriction to any local algebra. This fact allows to extend the representa-

tions and endomorphisms to a net of local von Neumann algebras which is obtained by weak closure of local C^*-algebras in the vacuum sector. It is shown that the net obeys Haag duality on the punctured circle. Thus we are in the position that the composition of our special localized endomorphisms generalizes indeed to their unitary equivalence classes, i.e. we have a well-defined DHR sector product, and we prove rigorously that it reproduces the WZW fusion rules.

We believe that the proof of the WZW fusion rules in terms of the DHR sector product is remarkable because it is completely independent of the methods that are used in CFT to derive fusion rules (and which do not all possess a mathematically satisfactory status). Therefore we believe the goal of this analysis to be two-fold: Firstly, a non-trivial model of CFT could be incorporated into the DHR framework and expected correspondences between CFT and AQFT could be established explicitly. Secondly, the conformal fusion rules could be proven in a rather independent way.

1.4.2 Results (Level 2)

It is well-known that $k^\vee$-fold tensor products of level 1 modules contain level $k^\vee$ modules (with an infinite multiplicity) and that all level $k^\vee$ modules can be realized in this way. More precisely, the tensor products of $\mathfrak{g}_1$-modules decompose into tensor products of $\mathfrak{g}_{k^\vee}$-modules and modules of the coset CFT $(\mathfrak{g}_1)^{\oplus k^\vee}/\mathfrak{g}_{k^\vee}$. We employ this idea to realize sectors of $\widehat{\mathfrak{so}}(N)$ WZW models at level 2: In the tensor product space $\hat{\mathcal{H}}_{\mathrm{NS}} = \mathcal{H}_{\mathrm{NS}} \otimes \mathcal{H}_{\mathrm{NS}}$ (the "big Fock space") we have a canonical action of $\widehat{\mathfrak{so}}(N)$ at level 2. Hence all the level 2 modules that appear in tensor products of the level 1 basic and vector modules are realized in the big Fock space $\hat{\mathcal{H}}_{\mathrm{NS}}$. It is somewhat surprising that, although we do not consider tensor products of the level 1 spinor modules, "nearly all" level 2 WZW sectors are realized in $\hat{\mathcal{H}}_{\mathrm{NS}}$; $\ell+3$ of $\ell+7$ if $N=2\ell$ respectively $\ell+2$ of $\ell+4$ if $N=2\ell+1$. However, since they appear with infinite multiplicities it would in general be a rather hopeless task to identify all irreducible components within $\hat{\mathcal{H}}_{\mathrm{NS}}$. It is in fact the DHR theory that yields a very useful order structure in the big Fock space: On $\hat{\mathcal{H}}_{\mathrm{NS}}$ their acts a canonical fermion field algebra $\mathfrak{F}$ which can be built out of a net of local field algebras. The fermionic expressions of the level 2 currents are invariant under the gauge group $O(2)$. Hence the full chiral algebra of the WZW models is gauge invariant. Moreover, the Virasoro generators of

the relevant coset CFT

$$\mathfrak{Cos} = (\widehat{\mathfrak{so}}(N)_1 \oplus \widehat{\mathfrak{so}}(N)_1)/\widehat{\mathfrak{so}}(N)_2$$

living in $\hat{\mathcal{H}}_{\rm NS}$ are gauge invariant as well. We introduce an algebra $\mathfrak{A}$ as the gauge invariant part of $\mathfrak{F}$ and we can decompose the big Fock space into sectors of the gauge invariant fermion algebra à la DHR. By gauge invariance, neither the currents of $\mathfrak{so}(N)_2$ nor the coset Virasoro operators can make transitions between these sectors. However, the $\mathfrak{A}$ sectors are not irreducible with respect to the action of $\widehat{\mathfrak{so}}(N)_2$. Indeed we have proper inclusions[6]

$$\mathfrak{A}_{\rm WZW} \subset \mathfrak{A} \subset \mathfrak{F}\,.$$

Here $\mathfrak{A}_{\rm WZW}$ denotes the algebra of bounded operators which corresponds to the chiral algebra of the WZW model: It can be constructed from the action of local loop groups $L_I G$ with $G = SO(N)$ in $\hat{\mathcal{H}}_{\rm NS}$. The algebra $\mathfrak{A}_{\rm WZW}$ contains the bounded functions of locally smeared currents and the irreducible spaces of the positive energy representations are precisely the highest weight modules of the chiral algebra.

Fortunately, the central charge of the Virasoro algebra of the coset CFT $(\widehat{\mathfrak{so}}(N)_1 \oplus \widehat{\mathfrak{so}}(N)_1)/\widehat{\mathfrak{so}}(N)_2$ equals one, and the $c = 1$ CFTs are all classified. In our context, it easily turns out to be a so-called $c = 1$ $\mathbb{Z}_2$ orbifold which is well understood. In particular, the characters of its irreducible modules are given in the literature. However, the branching rules corresponding to the embedding of $\widehat{\mathfrak{so}}(N)_2$ modules in the tensor products of $\widehat{\mathfrak{so}}(N)_1$ basic and vector modules were not completely known; they turn out as a result of our analysis. We present a large set of simultaneous highest weight vectors of $\widehat{\mathfrak{so}}(N)_2$ and the coset Virasoro algebra. Moreover, a detailed analysis of the Virasoro specialized characters corresponding to the sectors of $\mathfrak{A}$ allows to puzzle out the complete decomposition of $\hat{\mathcal{H}}_{\rm NS}$ into tensor products of $\widehat{\mathfrak{so}}(N)_2$ highest weight modules and irreducible modules of the coset Virasoro algebra. The solution of this puzzle includes also an independent derivation of the level 2 characters which are, however, already known in the literature. We believe that besides these results the explicit construction of level 2 modules and modules of the coset CFT within tensor products of level 1 modules

[6] That the inclusion $\mathfrak{A}_{\rm WZW} \subset \mathfrak{A}$ is proper is clear since at least the coset Virasoro algebra which does not belong to the WZW chiral algebra is gauge invariant as well. This is in contrast to the situation at level 1 where the gauge ($\mathbb{Z}_2$) invariant fermion algebra is identified to be the observable algebra of the WZW model.

is of some particular interest because all these structures become relatively transparent due to the fermionic realization.

1.4.3 Overview

This dissertation is organized as follows. In Chapter 2 we present two rather different mathematical tools which will be intensively employed in our analysis of the WZW models. Firstly, we introduce Araki's selfdual CAR algebra and present some well-known results on quasi-free states, Fock representations and Bogoliubov transformations. We have included some of our own results on the decomposition of representations induced by Bogoliubov endomorphisms into irreducibles, because these theorems will be applied in Chapter 3. Secondly, we present several technical details of the simple Lie algebra $\mathfrak{so}(N)$ and the associated affine Lie algebra $\widehat{\mathfrak{so}}(N)$ since this will become relevant in the following chapters, especially in Chapter 4. We also list the unitary highest weight modules and fusion rules of the $\mathfrak{so}(N)$ WZW models at level 1 and 2.

In Chapter 3 we analyze the $\mathfrak{so}(N)$ WZW models at level 1. We describe how the realizations of $\widehat{\mathfrak{so}}(N)_1$ arise via second quantization from a natural action of $\widehat{\mathfrak{so}}(N)$ in the pre-quantized space $L^2(S^1;\mathbb{C}^N)$. Employing the representation theory of even CAR we discuss in detail the treatment of the WZW models in the algebraic framework: We introduce a net of local C^*–algebras, construct localized endomorphisms and extend them to a net of local von Neumann algebras. Finally we prove the WZW fusion rules in terms of the DHR sector product.

In Chapter 4 we analyze the $\mathfrak{so}(N)$ WZW models at level 2. We introduce the big Fock space and, acting on it, the fermion algebra and its gauge invariant subalgebra. Employing the representation theory of the gauge group $O(2)$ we decompose the big Fock space into the DHR sectors of the gauge invariant fermion algebra. We present a large set of simultaneous highest weight vectors of $\widehat{\mathfrak{so}}(N)_2$ and the coset Virasoro algebra $\mathfrak{Vir}^{\mathrm{c}}$ in terms of fermionic creation operators acting on the vacuum. A detailed analysis of the Virasoro specialized characters ends up with the decomposition of the big Fock space into tensor products of level 2 highest weight modules and irreducible modules of the coset Virasoro algebra.

Chapter 2

Mathematical Preliminaries

In this chapter we prepare some mathematical background because it will be needed in the following chapters. In order to keep these mathematical technicalities out of the later discussion of the WZW models, we treat two rather different mathematical fields here. Firstly, we discuss the CAR algebra, quasi-free states and Bogoliubov transformations in Araki's selfdual formulation of CAR. Secondly, we treat the Lie algebras $\mathfrak{so}(N)$ and $\widehat{\mathfrak{so}}(N)$.

2.1 The Selfdual CAR Algebra

We follow essentially Araki's articles [1, 2]. For a detailed discussion of the CAR algebra and the connections between its different formulations see also [22].

2.1.1 Basic Notations and Results

Let $\mathcal{K}$ be some infinite dimensional separable Hilbert space endowed with an antiunitary involution Γ (complex conjugation), $\Gamma^2 = \mathbf{1}$, which obeys

$$\langle \Gamma f, \Gamma g \rangle = \langle g, f \rangle \,, \qquad f, g \in \mathcal{K} \,.$$

The selfdual CAR algebra $\mathcal{C}(\mathcal{K}, \Gamma)$ is defined to be the C^*-norm closure of the algebra which is generated by the range of a linear mapping $B : f \mapsto B(f)$, such that

$$\{B(f)^*, B(g)\} = \langle f, g \rangle \mathbf{1} \,, \qquad B(f)^* = B(\Gamma f) \,, \qquad f, g \in \mathcal{K} \,,$$

holds. The C^*-norm satisfies

$$\|B(f)\| \leq \|f\|, \qquad f \in \mathcal{K}. \tag{2.1}$$

The states of $\mathcal{C}(\mathcal{K}, \Gamma)$ we are interested in are called quasi-free states. By definition, a quasi-free state ω fulfills for $n \in \mathbb{N}$

$$\begin{aligned} \omega(B(f_1) \cdots B(f_{2n+1})) &= 0, \\ \omega(B(f_1) \cdots B(f_{2n})) &= (-1)^{n(n-1)/2} \sum_{\sigma} \mathrm{sign}\sigma \prod_{j=1}^{n} \omega(B(f_{\sigma(j)})B(f_{\sigma(n+j)})) \end{aligned}$$

where the sum runs over all permutations $\sigma \in \mathcal{S}_{2n}$ with the property

$$\sigma(1) < \sigma(2) < \cdots < \sigma(n), \qquad \sigma(j) < \sigma(j+n), \qquad j = 1, 2, \ldots, n.$$

Clearly, quasi-free states are completely characterized by their two point functions. Moreover, there is a correspondence between the set of quasi-free states and the set

$$\mathcal{Q}(\mathcal{K}, \Gamma) = \{ S \in \mathfrak{B}(\mathcal{K}) \,|\, S = S^*,\, 0 \leq S \leq \mathbf{1},\, S + \overline{S} = \mathbf{1} \},$$

(we have used the notation $\overline{A} = \Gamma A \Gamma$ for operators A on $\mathcal{K}$) given by the formula

$$\omega(B(f)^* B(g)) = \langle f, Sg \rangle. \tag{2.2}$$

So it is convenient to denote the quasi-free state characterized by Eq. (2.2) by ω_S. The projections in $\mathcal{Q}(\mathcal{K}, \Gamma)$ are called basis projections or polarizations. For a basis projection P, the state ω_P is pure and is called a Fock state. The corresponding GNS representation $(\mathcal{H}_P, \pi_P, |\Omega_P\rangle)$ is irreducible, it is called a Fock representation. The space $\mathcal{H}_P$ can be canonically identified with the antisymmetric Fock space $\mathcal{F}_-(P\mathcal{K})$.

A projection $E \in \mathfrak{B}(\mathcal{K})$ with the property that $E\overline{E} = 0$ is called a partial basis projection with Γ-codimension $\dim\ker(E + \overline{E})$. Note that E defines a Fock representation $(\mathcal{H}_E, \pi_E, |\Omega_E\rangle)$ of $\mathcal{C}((E + \overline{E})\mathcal{K}, \Gamma)$. Now let E be a partial basis projection with Γ-codimension 1, and choose a Γ-invariant unit vector $e_0 \in \ker(E + \overline{E})$. Following Araki, pseudo Fock representations $\pi_{E,+}$ and $\pi_{E,-}$ of $\mathcal{C}(\mathcal{K}, \Gamma)$ are defined in $\mathcal{H}_E$ by

$$\pi_{E,\pm}(B(f)) = \pm \frac{1}{\sqrt{2}} \langle e_0, f \rangle Q_E(-\mathbf{1}) + \pi_E(B((E + \overline{E})f), \qquad f \in \mathcal{K}, \tag{2.3}$$

where $Q_E(-\mathbf{1}) \in \mathfrak{B}(\mathcal{K})$ is the unitary, self-adjoint implementer of the automorphism α_{-1} of $\mathcal{C}(\mathcal{K},\Gamma)$ defined by $\alpha_{-1}(B(f)) = -B(f)$ (which restricts also to an automorphism of $\mathcal{C}((E+\overline{E})\mathcal{K},\Gamma)$). Pseudo Fock representations $\pi_{E,+}$ and $\pi_{E,-}$ are inequivalent and irreducible. Araki proved [1]

Lemma 2.1 *Let E be a partial basis projection with Γ-codimension 1, and let $e_0 \in \mathcal{K}$ be a Γ-invariant unit vector of* $\ker(E+\overline{E})$. *Define $S \in \mathcal{Q}(\mathcal{K},\Gamma)$ by*

$$S = \frac{1}{2}\,|e_0\rangle\langle e_0| + E. \tag{2.4}$$

Then a GNS representation $(\mathcal{H}_S, \pi_S, |\Omega_S\rangle)$ of the quasi-free state ω_S is given by the direct sum of two inequivalent, irreducible pseudo Fock representations,

$$(\mathcal{H}_S, \pi_S, |\Omega_S\rangle) = \Big(\mathcal{H}_E \oplus \mathcal{H}_E, \pi_{E,+} \oplus \pi_{E,-}, \frac{1}{\sqrt{2}}\,(|\Omega_E\rangle \oplus |\Omega_E\rangle)\Big). \tag{2.5}$$

There is an important quasi-equivalence criterion for GNS representations of quasi-free states. Quasi-equivalence will be denoted by "$\approx$" and unitary equivalence by "$\simeq$". Let us denote by $\mathfrak{J}_2(\mathcal{K})$ the ideal of Hilbert-Schmidt operators in $\mathfrak{B}(\mathcal{K})$ and for $A \in \mathfrak{B}(\mathcal{K})$ by $[A]_2$ its Hilbert-Schmidt equivalence class $[A]_2 = A + \mathfrak{J}_2(\mathcal{K})$. Araki proved [1, 2]

Theorem 2.2 *For quasi-free states ω_{S_1} and ω_{S_2} of $\mathcal{C}(\mathcal{K},\Gamma)$ we have quasi-equivalence $\pi_{S_1} \approx \pi_{S_2}$ if and only if $[S_1^{1/2}]_2 = [S_2^{1/2}]_2$.*

Next we define the set

$$\mathcal{I}(\mathcal{K},\Gamma) = \{V \in \mathfrak{B}(\mathcal{K}) \,|\, V^*V = \mathbf{1},\ V = \overline{V}\}$$

of Bogoliubov operators. Bogoliubov operators $V \in \mathcal{I}(\mathcal{K},\Gamma)$ induce unital $*$-endomorphisms ϱ_V of $\mathcal{C}(\mathcal{K},\Gamma)$, defined by their action on the canonical generators,

$$\varrho_V(B(f)) = B(Vf).$$

Moreover, if $V \in \mathcal{I}(\mathcal{K},\Gamma)$ is surjective (i.e. unitary), then ϱ_V is an automorphism. The group of unitary Bogoliubov operators is denoted by $\mathcal{U}(\mathcal{K},\Gamma)$. A Bogoliubov automorphism ϱ_U is called implementable in a Fock representation π_P if there exists a unitary $Q_P(U) \in \mathfrak{B}(\mathcal{H}_P)$ such that

$$\pi_P \circ \varrho_U(x) = Q_P(U)\pi_P(x)Q_P(U)^*, \qquad x \in \mathcal{C}(\mathcal{K},\Gamma). \tag{2.6}$$

For a given basis projection P, let $\mathcal{U}_P(\mathcal{K},\Gamma)$ be the set of unitary Bogoliubov operators which induce automorphisms being implementable in π_P,

$$\mathcal{U}_P(\mathcal{K},\Gamma) = \{U \in \mathcal{U}(\mathcal{K},\Gamma) \mid PU\overline{P} \in \mathfrak{J}_2(\mathcal{K})\}. \tag{2.7}$$

$\mathcal{U}_P(\mathcal{K},\Gamma)$ is a topological group with respect to the P-norm topology, given by the distance $\|U-U'\| + \|P(U-U')\overline{P}\|_2$. The P-strong topology is defined by the seminorms $\|(U-U')f\| + \|P(U-U')\overline{P}\|_2$, $f \in \mathcal{K}$. Consider the (real) Lie algebra

$$\mathfrak{u}_P^{\mathrm{b}}(\mathcal{K},\Gamma) = \{H \in \mathfrak{B}(\mathcal{K}) \mid H^* = -H,\ H = \overline{H},\ PH\overline{P} \in \mathfrak{J}_2(\mathcal{K})\}. \tag{2.8}$$

The following is well known (see e.g. [2, 13])

Theorem 2.3 *(1) U_t is a continuous one-parameter subgroup of $\mathcal{U}_P(\mathcal{K},\Gamma)$ relative to the P-norm topology if and only if $U_t = \mathrm{e}^{tH}$ for some $H \in \mathfrak{u}_P^{\mathrm{b}}(\mathcal{K},\Gamma)$.*

(2) For $H \in \mathfrak{u}_P^{\mathrm{b}}(\mathcal{K},\Gamma)$ there is a unique skew self-adjoint $\mathrm{d}Q_P(H)$ on $\mathcal{H}_P$ such that the unitary $\mathrm{e}^{t\mathrm{d}Q_P(H)}$ implements e^{tH}, $|\Omega_P\rangle$ is in its domain and $\langle\Omega_P|\mathrm{d}Q_P(H)|\Omega_P\rangle = 0$. The map $H \mapsto \mathrm{d}Q_P(H)$ is real linear.

(3) Any vector in the (finite) linear span of finite particle vectors is analytic for $-\mathrm{i}\,\mathrm{d}Q_P(H)$ for any $H \in \mathfrak{u}_P^{\mathrm{b}}(\mathcal{K},\Gamma)$ and we have

$$[\mathrm{d}Q_P(H_1), \mathrm{d}Q_P(H_2)] = \mathrm{d}Q_P([H_1,H_2]) + c_P(H_1,H_2)\,\mathbf{1}, \tag{2.9}$$

where

$$c_P(H_1,H_2) = \tfrac{1}{2}\,\mathrm{tr}\,(PH_2\overline{P}H_1P - PH_1\overline{P}H_2P). \tag{2.10}$$

(4) For $H \in \mathfrak{u}_P^{\mathrm{b}}(\mathcal{K},\Gamma)$, $\mathrm{e}^{t\mathrm{d}Q_P(H)}$ is strongly continuous with respect to the P-strong topology.

The scalar term (2.10) which appears here is called the "Schwinger term". There is also a generalization to the case that H is unbounded [2].

Theorem 2.4 *(1) U_t is a continuous one-parameter subgroup of $\mathcal{U}_P(\mathcal{K},\Gamma)$ relative to the P-strong topology if $U_t = \mathrm{e}^{tH}$ for a skew self-adjoint operator H such that $H = \overline{H}$, PHP skew self-adjoint and the closure of $\overline{P}HP$ is Hilbert-Schmidt class.*

(2) For such an H, there is a unique skew self-adjoint $\mathrm{d}Q_P(H)$ on $\mathcal{H}_P$ such that the unitary $\mathrm{e}^{t\mathrm{d}Q_P(H)}$ implements e^{tH}, $|\Omega_P\rangle$ is in its domain and $\langle\Omega_P|\mathrm{d}Q_P(H)|\Omega_P\rangle = 0$.

We conclude that $\mathfrak{u}_P^{\mathrm{b}}(\mathcal{K},\Gamma)$ is just the "bounded part" of the Lie algebra corresponding to the group $\mathcal{U}_P(\mathcal{K},\Gamma)$.

2.1.2 Decomposition of Representations

We now investigate the decomposition of representations $\pi_P \circ \varrho_V$ into irreducible subrepresentations; here π_P is a Fock representation and ϱ_V a Bogoliubov endomorphism. The discussion is based on [6, 7], compare also [4].

A quasi-free state, composed with a Bogoliubov endomorphism is again a quasi-free state, namely we have $\omega_S \circ \varrho_V = \omega_{V^*SV}$. In the following we are interested in representations of the form $\pi_P \circ \varrho_V$ instead of GNS representations π_{V^*PV} of states $\omega_{V^*PV} = \omega_P \circ \varrho_V$. Indeed, the former are multiples of the latter, in particular, we have [4, 48]

$$\pi_P \circ \varrho_V \simeq 2^{N_V} \pi_{V^*PV}, \qquad N_V = \dim(\ker V^* \cap P\mathcal{K}). \tag{2.11}$$

Thus the identification of the Hilbert-Schmidt equivalence class $[(V^*PV)^{1/2}]_2$ is the identification of the quasi-equivalence class of $\pi_P \circ \varrho_V$. For the identification of the unitary equivalence class, we need a decomposition of $\pi_P \circ \varrho_V$ into irreducible subrepresentations which will now be elaborated. We will see that only Fock and pseudo Fock representations appear in the decomposition of representations $\pi_P \circ \varrho_V$ if the Bogoliubov operator has finite corank.

Theorem 2.5 *Let P be a basis projection and let V be a Bogoliubov operator with $M_V = \dim \ker V^* < \infty$. If M_V is an even integer we have (with notations as above)*

$$\pi_P \circ \varrho_V \simeq 2^{M_V/2} \pi_{P'} \tag{2.12}$$

where $\pi_{P'}$ is an (irreducible) Fock representation. If M_V is odd then we have

$$\pi_P \circ \varrho_V \simeq 2^{(M_V-1)/2} (\pi_{E,+} \oplus \pi_{E,-}) \tag{2.13}$$

where $\pi_{E,+}$ and $\pi_{E,-}$ are inequivalent (irreducible) pseudo Fock representations.

Proof. We present a brief version of the proofs in [6] here. First recall the following well-known fact. If the test function space possesses an orthogonal decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ into Γ-invariant subspaces then we can regard $\mathcal{C}(\mathcal{K}_j, \Gamma_j)$ with $\Gamma_j = \Gamma|_{\mathcal{K}_j}$, $j = 1, 2$, as subalgebras of $\mathcal{C}(\mathcal{K}, \Gamma)$, and $\mathcal{C}(\mathcal{K}, \Gamma)$ is canonically isomorphic to the $\mathbb{Z}_2$-graded tensor product of $\mathcal{C}(\mathcal{K}_1, \Gamma_1)$ with $\mathcal{C}(\mathcal{K}_2, \Gamma_2)$ by the identification $x_1 \otimes x_2$ with $x_1 x_2$, $x_j \in \mathcal{C}(\mathcal{K}_j, \Gamma_j)$, $j = 1, 2$. Let $S_j \in \mathcal{Q}(\mathcal{K}_j, \Gamma_j)$, $j = 1, 2$. If $S \in \mathcal{Q}(\mathcal{K}, \Gamma)$ is of the form $S = S_1 \oplus S_2$ then ω_S is

a product state, $\omega_S = \omega_{S_1} \otimes \omega_{S_2}$ which means that $\omega_S(x_1 x_2) = \omega_{S_1}(x_1)\omega_{S_2}(x_2)$ whenever $x_j \in \mathcal{C}(\mathcal{K}_j, \Gamma_j)$, $j = 1, 2$, [42, 44]. In our application, $S_1 = P_1$ is a basis projection, and we assume $\mathcal{K}_2$ to be two-dimensional here. Then choose an orthonormal base (ONB) $\{f_+, f_-\}$ of $\mathcal{K}_2$ with the property $\Gamma_2 f_+ = f_-$. One checks easily that $S_2 \in \mathcal{Q}(\mathcal{K}_2, \Gamma_2)$ implies that it is of the form

$$S_2 = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}, \qquad 0 \le \lambda_\pm \le 1, \qquad \lambda_+ + \lambda_- = 1,$$

with respect to the decomposition $\mathcal{K}_2 = \mathbb{C}f_+ \oplus \mathbb{C}f_-$. Using the only non-trivial evaluation of ω_{S_2} on $B(f_+)B(f_-)$ it is obvious that

$$\omega_{S_2} = \lambda_+ \omega_{F_+} + \lambda_- \omega_{F_-}$$

with Fock states $\omega_{F_\pm}$ corresponding to basis projections $F_\pm = |f_\pm\rangle\langle f_\pm|$. Hence, if both λ_+ and λ_- are non-zero, then ω_S is a mixture of two Fock states of $\mathcal{C}(\mathcal{K}, \Gamma)$,

$$\omega_S = \lambda_+ \omega_{P_+} + \lambda_- \omega_{P_-}, \qquad P_\pm = P_1 \oplus F_\pm.$$

Since $\omega_{P_+} \neq \omega_{P_-}$ these states are orthogonal, and hence π_S is the direct sum of two equivalent Fock representations. Now let P be a basis projection of $\mathcal{K}$ and let $V \in \mathcal{I}(\mathcal{K}, \Gamma)$ be a Bogoliubov operator with $M_V = \dim\ker V^* = 2$. Choose an ONB $\{q_+, q_-\}$ of $\ker V^*$ such that $\Gamma q_+ = q_-$. Then $\langle q_+, P q_- \rangle = 0$. Let $\lambda_\pm = \langle q_\pm, P q_\pm \rangle$, hence $\lambda_+ + \lambda_- = 1$, and define $Q_\pm = |q_\pm\rangle\langle q_\pm|$. Note that $VV^* = \mathbf{1} - Q_+ - Q_-$. Suppose $\lambda_\pm \neq 0$. Then

$$P_1 = V^* P(\mathbf{1} - \lambda_+^{-1} Q_+ - \lambda_-^{-1} Q_-) P V$$

is a partial basis projection with Γ-codimension 2. Define one-dimensional projections

$$F_\pm = \lambda_+^{-1} \lambda_-^{-1} V^* P Q_\mp P V,$$

$F_+ = \overline{F_-}$, $P_1 F_\pm = F_+ F_- = 0$, and set

$$S_2 = \lambda_+ F_+ + \lambda_- F_-.$$

We identify $\mathcal{K}_1 = \operatorname{ran}(P_1 + \overline{P_1})$ and $\mathcal{K}_2 = \operatorname{ran}(F_+ + F_-)$ and check that $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$. Hence

$$S = V^* P V = P_1 + S_2$$

is as discussed above. Because $\lambda_\pm \neq 0$ we have $q_\pm \notin P\mathcal{K}$, hence $N_V = 0$ and $\pi_P \circ \varrho_V \simeq \pi_{V^*PV}$ by Eq. (2.11). So it follows $\pi_P \circ \varrho_V \simeq 2\pi_{P'}$ with some Fock representation $\pi_{P'}$. On the other hand, if $\lambda_- = 0$ ($\lambda_+ = 0$) then $q_+ \in P\mathcal{K}$ ($q_- \in P\mathcal{K}$) and hence $N_V = 1$. It follows $\pi_P \circ \varrho_V \simeq 2\pi_{V^*PV}$ by Eq. (2.11). But $q_+ \in P\mathcal{K}$ ($q_- \in P\mathcal{K}$) implies that V^*PV is a projection and hence π_{V^*PV} is a Fock representation. We conclude that $\pi_P \circ \varrho_V \simeq 2\pi_{P'}$ holds generally if $M_V = 2$. Let now $M_V = 2N$, $N \in \mathbb{N}$. Then choose a Γ-invariant ONB $\{v_n, n \in \mathbb{N}\}$ of $\mathcal{K}$, i.e. $v_n = \Gamma v_n$, $n \in \mathbb{N}$. Further we choose a Γ-invariant ONB $\{w_n, n = 1, 2, \dots, 2N\}$ of $\ker V^*$, and we define $w_{2N+n} = V v_n$ for $n \in \mathbb{N}$. Since $\mathcal{K} = \mathrm{ran} V \oplus \ker V^*$ the set $\{w_n, n \in \mathbb{N}\}$ forms another Γ-invariant ONB of $\mathcal{K}$ and we can write

$$V = \sum_{n=1}^{\infty} |w_{2N+n}\rangle\langle v_n|.$$

We introduce Bogoliubov operators $V_0, V_2 \in \mathcal{I}(\mathcal{K}, \Gamma)$,

$$V_0 = \sum_{n=1}^{\infty} |w_n\rangle\langle v_n|, \qquad V_2 = \sum_{n=1}^{\infty} |v_{n+2}\rangle\langle v_n|,$$

such that $M_{V_j} = j$, $j = 0, 2$, and $V = V_0 V_2^N$. Since V_0 is unitary, $P_0 = V_0^* P V_0$ is again a basis projection, and now we can conclude iteratively

$$\pi_P \circ \varrho_V = \pi_{P_0} \circ \varrho_{V_2}^N \simeq 2^N \pi_{P'}$$

with some Fock representation $\pi_{P'}$, and this is Eq. (2.12). Now assume $V \in \mathcal{I}(\mathcal{K}, \Gamma)$ has $M_V = 1$. Since $\ker V^*$ is Γ-invariant we have $N_V = 0$ and hence $\pi_P \circ \varrho_V \simeq \pi_{V^*PV}$ by Eq. (2.11). Moreover, $VV^* = \mathbf{1} - Q_0$ where Q_0 is a one-dimensional Γ-invariant projection. Note that $\langle q_0, P q_0\rangle = \frac{1}{2}$ for each Γ-invariant unit vector q_0. Using $Q_0 P Q_0 = \frac{1}{2} Q_0$ one finds easily

$$S = V^*PV = \frac{1}{2}\, E_0 + E$$

where $E = 2S^2 - S$ is a partial basis projection with Γ-codimension 1, and $E_0 = 4(S - S^2) = 4V^*PQ_0PV$ is the one-dimensional Γ-invariant projection on the vector V^*Pq_0, $EE_0 = \overline{E}E_0 = 0$, thus $S = V^*PV$ is of the form (2.4) and hence

$$\pi_P \circ \varrho_V \simeq \pi_{E,+} \oplus \pi_{E,-}$$

with pseudo Fock representations $\pi_{E,\pm}$. Now let $V \in \mathcal{I}(\mathcal{K}, \Gamma)$ be a Bogoliubov operator with $M_V = 2N+1$, $N \in \mathbb{N}$. Recall our Γ-invariant ONB $\{v_n, n \in \mathbb{N}\}$

of $\mathcal{K}$ and choose a Γ-invariant ONB $\{w_n, n = 0, 1, \dots, 2N\}$ of $\ker V^*$. We define $w_{2N+n} = Vv_n$ for $n \in \mathbb{N}$. Then $\{w_n, n \in \mathbb{N}_0\}$ is an ONB of $\mathcal{K}$, too, and we can write

$$V = \sum_{n=1}^{\infty} |w_{2N+n}\rangle\langle v_n|.$$

We introduce Bogoliubov operators $V_1, V_2 \in \mathcal{I}(\mathcal{K}, \Gamma)$,

$$V_1 = \sum_{n=1}^{\infty} |w_n\rangle\langle v_n|, \qquad V_2 = \sum_{n=0}^{\infty} |w_{n+2}\rangle\langle w_n|,$$

such that $M_{V_j} = j$, $j = 1, 2$, and $V = V_2^N V_1$. It follows that

$$\pi_P \circ \varrho_V \simeq 2^N \pi_{P'} \circ \varrho_{V_1}$$

with $\pi_{P'}$ some Fock representation. Since $M_{V_1} = 1$ we conclude

$$\pi_P \circ \varrho_V \simeq 2^N (\pi_{E,+} \oplus \pi_{E,-})$$

with pseudo Fock representations $\pi_{E,\pm}$, and this is Eq. (2.13). □

We define the even algebra $\mathcal{C}(\mathcal{K}, \Gamma)^+$ to be the subalgebra of α_{-1}-fixpoints,

$$\mathcal{C}(\mathcal{K}, \Gamma)^+ = \{x \in \mathcal{C}(\mathcal{K}, \Gamma) \mid \alpha_{-1}(x) = x\}.$$

We now are interested in what happens when our representations of $\mathcal{C}(\mathcal{K}, \Gamma)$ are restricted to the even algebra. For basis projections P_1, P_2, with $[P_1]_2 = [P_2]_2$, Araki and D.E. Evans [3] defined an index, taking values ± 1,

$$\operatorname{ind}(P_1, P_2) = (-1)^{\dim(P_1\mathcal{K} \cap (\mathbf{1} - P_2)\mathcal{K})}.$$

The automorphism α_{-1} leaves any quasi-free state ω_S invariant. Hence α_{-1} is implemented in π_S. In particular, in a Fock representation π_P, α_{-1} extends to an automorphism $\bar{\alpha}_{-1}$ of $\pi_P(\mathcal{C}(\mathcal{K}, \Gamma))'' = \mathfrak{B}(\mathcal{H}_P)$. The following proposition is taken from [2].

Proposition 2.6 *Let P be a basis projection and let $U \in \mathcal{U}_P(\mathcal{K}, \Gamma)$. Denote by $Q_P(U) \in \mathfrak{B}(\mathcal{H}_P)$ the unitary which implements ϱ_U in π_P. Then*

$$\bar{\alpha}_{-1}(Q_P(U)) = \sigma(U) Q_P(U), \qquad \sigma(U) = \pm 1. \tag{2.14}$$

In particular, $\sigma(U) = \operatorname{ind}(P, U^*PU)$. Moreover, given two unitary operators $U_1, U_2 \in \mathcal{U}_P(\mathcal{K}, \Gamma)$ of this type, σ is multiplicative, $\sigma(U_1U_2) = \sigma(U_1)\sigma(U_2)$.

Furthermore, one has [3, 2]

Theorem 2.7 *Restricted to the even algebra $\mathcal{C}(\mathcal{K},\Gamma)^+$, a Fock representation π_P splits into two mutually inequivalent, irreducible subrepresentations,*

$$\pi_P|_{\mathcal{C}(\mathcal{K},\Gamma)^+} = \pi_P^+ \oplus \pi_P^-, \tag{2.15}$$

and the commutant is generated by $Q_P(-\mathbf{1})$. Given two basis projections P_1, P_2, then $\pi_{P_1}^{\pm} \simeq \pi_{P_2}^{\pm}$ if and only if $[P_1]_2 = [P_2]_2$ and $\operatorname{ind}(P_1,P_2) = +1$, and $\pi_{P_1}^{\pm} \simeq \pi_{P_2}^{\mp}$ if and only if $[P_1]_2 = [P_2]_2$ and $\operatorname{ind}(P_1,P_2) = -1$.

For some real $v \in \mathcal{K}$, i.e. $\Gamma v = v$, and $\|v\| = 1$ define $U \in \mathcal{U}(\mathcal{K},\Gamma)$ by

$$U = 2|v\rangle\langle v| - \mathbf{1}. \tag{2.16}$$

Then ϱ_U is implemented in each Fock representation π_P by the unitary self-adjoint $Q_P(U) = \sqrt{2}\pi_P(B(v))$, since ϱ_U is implemented in $\mathcal{C}(\mathcal{K},\Gamma)$ by $q(U) = \sqrt{2}B(v)$,

$$\begin{aligned}
q(U)B(f)q(U) &= 2B(v)B(f)B(v) \\
&= 2\{B(v), B(f)\}B(v) - 2B(f)B(v)B(v) \\
&= 2\langle v, f\rangle B(v) - B(f) \\
&= B(2\langle v, f\rangle v - f) \\
&= B(Uf).
\end{aligned}$$

Hence $\sigma(U) = -1$ and we immediately have the following

Corollary 2.8 *Let $U \in \mathcal{I}(\mathcal{K},\Gamma)$ be as in Eq. (2.16). Then, in restriction to $\mathcal{C}(\mathcal{K},\Gamma)^+$, we have for each Fock representation π_P equivalence $\pi_P^{\pm} \circ \varrho_U \simeq \pi_P^{\mp}$.*

Now let us consider the restrictions of pseudo Fock representations.

Lemma 2.9 *The pseudo Fock representations $\pi_{E,+}$ and $\pi_{E,-}$ of Eq. (2.3), when restricted to the even algebra $\mathcal{C}(\mathcal{K},\Gamma)^+$, remain irreducible and become equivalent.*

Proof. Without loss of generality, we prove that $\pi_{E,+}$, when restricted to $\mathcal{C}(\mathcal{K},\Gamma)^+$, remains irreducible. Let $T \in \pi_{E,+}(\mathcal{C}(\mathcal{K},\Gamma)^+)'$. Then, in particular,

$T \in \pi_E(\mathcal{C}((E+\overline{E})\mathcal{K},\Gamma)^+)'$, hence $T = \lambda\mathbf{1} + \mu Q_E(-\mathbf{1})$, $\lambda, \mu \in \mathbb{C}$. Now choose a non-zero $f \in (E+\overline{E})\mathcal{K}$. Then

$$\pi_{E,+}(B(e_0)B(f)) = \frac{1}{\sqrt{2}} Q_E(-\mathbf{1})\pi_E(B(f)),$$

so we compute

$$[T, \pi_{E,+}(B(e_0)B(f))] = \sqrt{2}\mu\pi_E(B(f)).$$

This implies $\mu = 0$, $T = \lambda\mathbf{1}$, proving irreducibility. It remains to be shown that $\pi_{E,+}$ and $\pi_{E,-}$, when restricted to $\mathcal{C}(\mathcal{K},\Gamma)^+$, become equivalent. Now choose arbitrary $f, g \in \mathcal{K}$. It is not hard to check that

$$\pi_{E,+}(B(f)B(g)) = Q_E(-\mathbf{1})\pi_{E,-}(B(f)B(g))Q_E(-\mathbf{1}).$$

Since $\mathcal{C}(\mathcal{K},\Gamma)^+$ is generated by such elements $B(f)B(g)$ the unitary $Q_E(-\mathbf{1})$ realizes the equivalence of the restrictions of $\pi_{E,\pm}$. $\square$

Summarizing we obtain

Theorem 2.10 *With notations of Theorem 2.5, a representation $\pi_P \circ \varrho_V$ restricts as follows to the even algebra $\mathcal{C}(\mathcal{K},\Gamma)^+$: If M_V is even we have*

$$\pi_P \circ \varrho_V|_{\mathcal{C}(\mathcal{K},\Gamma)^+} \simeq 2^{M_V/2}(\pi^+_{P'} \oplus \pi^-_{P'}) \tag{2.17}$$

with $\pi^+_{P'}$ and $\pi^-_{P'}$ mutually inequivalent and irreducible. If M_V is odd, then

$$\pi_P \circ \varrho_V|_{\mathcal{C}(\mathcal{K},\Gamma)^+} \simeq 2^{(M_V+1)/2}\,\pi \tag{2.18}$$

with π irreducible.

2.2 On $\mathfrak{so}(N)$ and $\widehat{\mathfrak{so}}(N)$

We now turn to the mathematics of Lie algebras. In view of applications in the following chapters we introduce some notation and present some technical details of the simple Lie algebra $\mathfrak{so}(N)$ and the associated affine Lie algebra $\widehat{\mathfrak{so}}(N)$ because we believe that it might be somewhat laborious for the reader to collect these facts from the literature (e.g. [37, 43]).

2.2.1 The Simple Lie Algebra $\mathfrak{so}(N)$

Let $E^{i,j}$ be the $(N \times N)$-matrix with entries $(E^{i,j})_{k,l} = \delta_{i,k}\delta_{j,l}$. Define

$$T^{i,j} = \mathrm{i}\,(E^{i,j} - E^{j,i})$$

for $i, j = 1, 2, \dots, N$. Elements $T^{i,j}$, $1 \le i < j \le N$, provide a basis of $\mathfrak{so}(N)$, and we have

$$[T^{i,j}, T^{k,l}] = \mathrm{i}\,(\delta_{j,k}T^{i,l} + \delta_{i,l}T^{j,k} - \delta_{j,l}T^{i,k} - \delta_{i,k}T^{j,l}).$$

Let $\ell = 3, 4, \dots$ denote the rank of $\mathfrak{so}(N)$, i.e. $N = 2\ell$ and $N = 2\ell + 1$ for even and odd N, respectively. Define

$$t^{i,j}_{\varepsilon,\eta} = \tfrac{1}{2}\,(\varepsilon T^{2i,2j-1} + \eta T^{2i-1,2j}) + \tfrac{\mathrm{i}}{2}\,(T^{2i-1,2j-1} - \varepsilon\eta T^{2i,2j})$$

for $i, j = 1, 2, \dots, \ell$ and $\varepsilon, \eta = \pm 1$ and

$$t^{j}_{\varepsilon} = -\frac{1}{\sqrt{2}}\,(\varepsilon T^{2j-1,2\ell+1} - \mathrm{i}T^{2j,2\ell+1})$$

for $j = 1, 2, \dots, \ell$ and $\varepsilon = \pm 1$. Further define

$$\begin{aligned} H^j &= T^{2j-1,2j} && \text{for } j = 1, 2, \dots, \ell\,, \\ E^j_\pm &= \pm t^{j,j+1}_{\pm,\mp} && \text{for } j = 1, 2, \dots, \ell - 1\,, \end{aligned}$$

and

$$E^\ell_\pm = \begin{cases} \pm t^{\ell-1,\ell}_{\pm,\pm} & \text{for } N = 2\ell\,, \\ \pm t^{\ell}_{\pm} & \text{for } N = 2\ell + 1\,. \end{cases}$$

These matrices obey the commutation relations

$$\begin{aligned} [H^j, H^k] &= 0\,, \\ [H^j, E^k_\pm] &= \pm\,(\alpha^{(k)})^j\, E^k_\pm\,, \\ [E^j_+, E^k_-] &= \delta_{j,k}\, H^j\,, \end{aligned}$$

for $j, k = 1, 2, \ldots, \ell$, with

$$(\alpha^{(k)})^j = \delta_{j,k} - \delta_{j,k+1}$$

for $k = 1, 2, \ldots, \ell - 1$ and

$$(\alpha^{(\ell)})^j = \begin{cases} \delta_{j,\ell-1} + \delta_{j,\ell} & \text{for } N = 2\ell\,, \\ \delta_{j,\ell} & \text{for } N = 2\ell + 1\,. \end{cases}$$

Moreover, they also obey the Serre relations of $\mathfrak{so}(N)$. Hence they constitute a Cartan-Weyl basis of $\mathfrak{so}(N)$, and the $\alpha^{(k)}$ are the simple roots of $\mathfrak{so}(N)$. The elements corresponding to positive roots are $t^{i,j}_{+,-}$ and $t^{i,j}_{+,+}$ with $1 \le i < j \le \ell$, additionally t^j_+, $j = 1, 2, \ldots, \ell$, if $N = 2\ell + 1$; the one corresponding to the highest root θ is $E_\theta = t^{1,2}_{+,+}$.

Also note that the invariant bilinear form on $\mathfrak{so}(N)$ is

$$(T^{i,j}|T^{k,l}) = \tfrac{1}{2}\,\mathrm{tr}\,(T^{i,j}T^{k,l}) = \delta_{i,k}\delta_{j,l} - \delta_{i,l}\delta_{j,k}\,. \tag{2.19}$$

In particular, we have

$$(H^i|H^j) = \delta_{i,j} = (E^i_+|E^j_-)\,, \qquad (E^i_\pm|E^j_\pm) = 0\,.$$

The fundamental weights $\Lambda_{(j)}$, $j = 1, 2, \ldots, \ell$, of $\mathfrak{so}(N)$ are defined by

$$(\alpha^{(j)}, \Lambda_{(k)}) = \begin{cases} \tfrac{1}{2}\delta_{k,\ell} & \text{for } \ j = \ell,\ N = 2\ell + 1, \\ \delta_{k,j} & \text{else,} \end{cases}$$

and its components (in the orthogonal base) are

$$\Lambda_{(j)} = (\underbrace{1, 1, \ldots, 1}_{j \text{ times}}, 0, 0, \ldots, 0)$$

for $j = 1, 2, \ldots, \ell - 2$ and also for $j = \ell - 1$ if $N = 2\ell + 1$, as well as

$$\Lambda_{(\ell-1)} = \frac{1}{2}\,(1, 1, \ldots, 1, 1, -1)$$

for $N = 2\ell$, and

$$\Lambda_{(\ell)} = \frac{1}{2}\,(1, 1, \ldots, 1, 1, 1)\,.$$

2.2.2 The Affine Lie Algebra $\widehat{\mathfrak{so}}(N)$

The infinite-dimensional Lie algebra $\widehat{\mathfrak{so}}(N)$ is, by definition, the algebra generated by elements $J^{i,j}_m$, $i,j = 1,2,\dots,N$ and a central element K satisfying relations

$$[J^{i,j}_m, J^{k,l}_n] = \mathrm{i}(\delta_{j,k}\, J^{i,l}_{m+n} + \delta_{i,l}\, J^{j,k}_{m+n} - \delta_{j,l}\, J^{i,k}_{m+n} - \delta_{i,k}\, J^{j,l}_{m+n}) + m\,\delta_{m,-n}(T^{i,j}|T^{j,i})K, \tag{2.20}$$

and $[J^{i,j}_m, K] = 0$. The elements $J^{i,j}_m$, $1 \le i < j \le N$, and K provide a basis of $\widehat{\mathfrak{so}}(N)$ as a vector space. Introducing one further element D ("derivation") obeying

$$[D, J^{i,j}_m] = m\, J^{i,j}_m, \qquad [D, K] = 0,$$

one obtains the full affine Lie algebra $D^{(1)}_\ell$ if $N = 2\ell$ respectively $B^{(1)}_\ell$ if $N = 2\ell + 1$. Note that via identification $J_m(T^{i,j}) \equiv J^{i,j}_m$ and defining $J_m(T)$ linear in $T \in \mathfrak{so}(N)$, $m \in \mathbb{Z}$, the relations (2.20) can also be written as follows,

$$[J_m(T), J_n(T')] = J_{m+n}([T, T']) + m\,\delta_{m,-n}(T|T')\,K,$$

and $[J_m(T), K] = 0$. Also the affine Lie algebra $\widehat{\mathfrak{so}}(N)$ possesses a Chevalley basis; the Cartan subalgebra generators are

$$\mathcal{H}^j = J^{2j-1,2j}_0$$

for $j = 1,2,\dots,\ell$, and the Chevalley generators are given by

$$\mathcal{E}^j_\pm = \pm J_0(t^{j,j+1}_{\pm,\mp}) \qquad \text{for } j = 1,2,\dots,\ell-1\,,$$

and

$$\mathcal{E}^\ell_\pm = \begin{cases} \pm J_0(t^{\ell-1,\ell}_{\pm,\pm}) & \text{for } N = 2\ell\,, \\ \pm J_0(t^\ell_\pm) & \text{for } N = 2\ell+1\,, \end{cases}$$

further

$$\mathcal{E}^0_\pm = \pm J_{\pm 1}(t^{1,2}_{\mp,\mp})\,.$$

for $i,j = 1,2,\dots,\ell$ and $\varepsilon,\eta = \pm 1$.

The unitary integrable highest weight modules of $\widehat{\mathfrak{so}}(N)$ at level 1 are listed in table 2.1. There Λ denotes the highest weight with respect to the horizontal subalgebra $\mathfrak{so}(N)$, Δ the conformal weight, and $\mathcal{D}$ the quantum dimension. In the first column we provide a "name" for the associated primary field of the relevant WZW theory; in the following chapters we will use

these names as labels for the irreducible highest weight modules, i.e. write $\mathcal{H}_\Lambda = \mathcal{H}_\circ$ for $\Lambda = 0$ etc., and analogously for other quantities such as characters. (We find it convenient to use identical names for some of the fields at level one and at level two; when required to avoid ambiguities in the notation, we will always also specify the level.)

Table 2.1: Unitary highest weight modules of $\widehat{\mathfrak{so}}(N)$ at level 1 for $N = 2\ell$ (left) and for $N = 2\ell+1$ (right).

field	Λ	Δ	$\mathcal{D}$
$\circ$	0	0	1
v	$\Lambda_{(1)}$	$\frac{1}{2}$	1
s	$\Lambda_{(\ell-1)}$	$\frac{N}{16}$	1
c	$\Lambda_{(\ell)}$	$\frac{N}{16}$	1

field	Λ	Δ	$\mathcal{D}$
$\circ$	0	0	1
v	$\Lambda_{(1)}$	$\frac{1}{2}$	1
σ	$\Lambda_{(\ell)}$	$\frac{N}{16}$	$\sqrt{2}$

In the tables we have separated the modules by a horizontal line into two classes. In the fermionic description, the modules in the first part are in the Neveu-Schwarz sector, while those in the second part are in the Ramond sector.

Let us now turn to the fusion rules of the WZW model based on $\widehat{\mathfrak{so}}(N)_1$. Since the basic module ($\circ$) always represents the unit of the fusion ring it is denoted by 1 in this context.

If $N = 2\ell$ the fusion rules[1] read

$$\begin{aligned} &\mathrm{v}\times\mathrm{v} = 1\,, \qquad\qquad \mathrm{v}\times\mathrm{s} = \mathrm{c}\,, \\ &\mathrm{s}\times\mathrm{s} = \mathrm{c}\times\mathrm{c} = \begin{cases} 1 & \text{for } \ell\in 2\mathbb{Z}\,, \\ \mathrm{v} & \text{for } \ell\in 2\mathbb{Z}+1\,, \end{cases} \\ &\mathrm{s}\times\mathrm{c} = \begin{cases} \mathrm{v} & \text{for } \ell\in 2\mathbb{Z}\,, \\ 1 & \text{for } \ell\in 2\mathbb{Z}+1\,. \end{cases} \end{aligned} \tag{2.21}$$

All sectors are simple i.e. have unit quantum dimension.

[1] The fusion rules which are not listed explicitly all follow from the commutativity and the associativity of the fusion product and from the fact that 1 is the unit of the fusion ring.

If $N = 2\ell + 1$ the fusion rules read

$$\mathrm{v} \times \mathrm{v} = 1, \qquad \sigma \times \mathrm{v} = \sigma, \qquad \sigma \times \sigma = 1 + \mathrm{v}. \tag{2.22}$$

Only the σ sector is not simple; indeed we have $\mathcal{D}_\sigma = \sqrt{2}$.

We are going to describe the situation at level 2 now. If $N = 2\ell$ ($N = 2\ell+1$) we have $\ell+7$ ($\ell+4$) integrable highest weight modules which are listed in the following tables 2.2, 2.3. Again we have separated the modules by a horizontal line into two classes. In the fermionic description, the modules in the first part appear in the "doubled" Neveu-Schwarz sector $\mathcal{H}_{\mathrm{NS}} \otimes \mathcal{H}_{\mathrm{NS}}$ while those in the second part involve the Ramond sector.

Table 2.2: Unitary highest weight modules of $\widehat{\mathfrak{so}}(N)$ at level 2 for $N = 2\ell$.

field	Λ	Δ	$\mathcal{D}$
$\circ$	0	0	1
v	$2\Lambda_{(1)}$	1	1
s	$2\Lambda_{(\ell-1)}$	$\frac{N}{8}$	1
c	$2\Lambda_{(\ell)}$	$\frac{N}{8}$	1
$\phi_{[j]}$	$\begin{cases} \Lambda_{(j)} & \text{for } j = 1, 2, \dots, \ell-2, \\ \Lambda_{(\ell-1)} + \Lambda_{(\ell)} & \text{for } j = \ell - 1 \end{cases}$	$\frac{j(N-j)}{2N}$	2
σ	$\Lambda_{(\ell-1)}$	$\frac{N-1}{16}$	$\sqrt{\ell}$
τ	$\Lambda_{(\ell)}$	$\frac{N-1}{16}$	$\sqrt{\ell}$
σ'	$\Lambda_{(1)} + \Lambda_{(\ell-1)}$	$\frac{N+7}{16}$	$\sqrt{\ell}$
τ'	$\Lambda_{(1)} + \Lambda_{(\ell)}$	$\frac{N+7}{16}$	$\sqrt{\ell}$

Omitting the twisted sectors $\sigma, \sigma', (\tau, \tau')$ all other sectors generate a fusion subring $\mathcal{R}^{(2)}_{\mathrm{NS}}$ of the full fusion ring $\mathcal{R}^{(2)}_{\mathrm{WZW}}$ of the WZW model based on $\widehat{\mathfrak{so}}(N)_2$. The fusion rules of this fusion subring are the following: For $N = 2\ell$

Table 2.3: Unitary highest weight modules of $\widehat{\mathfrak{so}}(N)$ at level 2 for $N=2\ell+1$.

field	Λ	Δ	$\mathcal{D}$
$\circ$	0	0	1
v	$2\Lambda_{(1)}$	1	1
$\phi_{[j]}$	$\begin{cases} \Lambda_{(j)} & \text{for } j=1,2,\ldots,\ell-1, \\ 2\Lambda_{(\ell)} & \text{for } j=\ell \end{cases}$	$\frac{j(N-j)}{2N}$	2
σ	$\Lambda_{(\ell-1)}$	$\frac{N-1}{16}$	$\sqrt{\ell}$
σ'	$\Lambda_{(1)}+\Lambda_{(\ell-1)}$	$\frac{N+7}{16}$	$\sqrt{\ell}$

we have

$$\begin{aligned} & \mathrm{v}\times\mathrm{v}=1\,, \qquad \mathrm{v}\times\mathrm{s}=\mathrm{c}\,, \\ & \mathrm{s}\times\mathrm{s}=\mathrm{c}\times\mathrm{c}=\begin{cases} 1 & \text{for } \ell\in 2\mathbb{Z}\,, \\ \mathrm{v} & \text{for } \ell\in 2\mathbb{Z}+1\,, \end{cases} \\ & \mathrm{s}\times\mathrm{c}=\begin{cases} \mathrm{v} & \text{for } \ell\in 2\mathbb{Z}\,, \\ 1 & \text{for } \ell\in 2\mathbb{Z}+1\,, \end{cases} \\ & \mathrm{v}\times\phi_{[j]}=\phi_{[j]}\,, \qquad \mathrm{s}\times\phi_{[j]}=\mathrm{c}\times\phi_{[j]}=\phi_{[\ell-j]}\,, \\ & \phi_{[i]}\times\phi_{[j]}=\phi_{[\,|i-j|\,]}+\phi_{[i+j]}\,. \end{aligned} \tag{2.23}$$

Here it is to be understood that whenever on the right hand side a label j appears which is larger than ℓ, it must be interpreted as the number

$$j'=N-j\,,$$

and when the label equals zero or ℓ, one has to identify $\phi_{[j]}$ as the sum

$$\phi_{[0]}\equiv 1+\mathrm{v}\,, \qquad \phi_{[\ell]}\equiv \mathrm{s}+\mathrm{c}\,.$$

For $N=2\ell+1$ the fusion rules read

$$\begin{aligned} & \mathrm{v}\times\mathrm{v}=1\,, \qquad \mathrm{v}\times\phi_{[j]}=\phi_{[j]}\,, \\ & \phi_{[i]}\times\phi_{[j]}=\phi_{[\,|i-j|\,]}+\phi_{[i+j]}\,. \end{aligned} \tag{2.24}$$

This time it is understood that when j is larger than ℓ, it stands for the number $j'=N-j$, and again that $\phi_{[0]}\equiv 1+\mathrm{v}$.

Chapter 3

$\mathfrak{so}(N)$ Wess-Zumino-Witten Models at Level 1

In this chapter we give a formulation of the level 1 $\mathfrak{so}(N)$ WZW models. We discuss the realization of $\widehat{\mathfrak{so}}(N) \rtimes \mathfrak{Vir}$ in the Neveu-Schwarz and in the Ramond sector. Employing the fact that the representation theory of the even fermion algebra reproduces the sectors of the chiral algebra, we can introduce a net of local C^*-algebras in terms of even CAR algebras and define endomorphisms that generate the WZW sectors. We extend their action to a net of local von Neumann algebras, and then we can prove the WZW fusion rules in terms of the DHR sector product.

This analysis is based on [7] and is a generalization of the program carried out for the Ising model [5]. As [5] was motivated from the earlier work [41] of Mack and Schomerus, [7] is motivated from the ideas of Fuchs, Ganchev and Vecsernyés [27].

3.1 Realization of $\widehat{\mathfrak{so}}(N) \rtimes \mathfrak{Vir}$

We begin our analysis with the fermionic realization of the level 1 modules coming from second quantization in the Neveu-Schwarz and the Ramond sector.

3.1.1 Representation in $\mathcal{K}$

From now on, let $\mathcal{K} = L^2(S^1;\mathbb{C}^N) \equiv L^2(S^1)\otimes\mathbb{C}^N$. We define a (Fourier) orthonormal base

$$\left\{e_r^i, \quad r\in\mathbb{Z}+\tfrac{1}{2}, \quad i=1,2,\dots,N\right\}$$

by the definition

$$e_r^i = e_r \otimes u^i$$

where $e_r \in L^2(S^1)$ are defined by $e_r(z) = z^r$, $z = \mathrm{e}^{\mathrm{i}\phi}$, $-\pi < \phi \leq \pi$, and u^i denote the canonical unit vectors of $\mathbb{C}^N$. Further we denote by Γ the canonical complex conjugation in $L^2(S^1;\mathbb{C}^N)$ so that $\Gamma e_r^i = e_{-r}^i$. We then define the Neveu-Schwarz operator $P_{\mathrm{NS}} \in \mathcal{Q}(\mathcal{K},\Gamma)$ to be the basis projection

$$P_{\mathrm{NS}} = \sum_{i=1}^{N}\sum_{r\in\mathbb{N}_0+1/2} |e_{-r}^i\rangle\langle e_{-r}^i|.$$

For $i,j = 1,2,\dots,N$ and $m\in\mathbb{Z}$, we define the following operators in $\mathfrak{B}(\mathcal{K})$,

$$\beta_m^{i,j} = \sum_{r\in\mathbb{Z}+1/2} |e_{r+m}^i\rangle\langle e_r^j|.$$

One checks by direct computation

$$[\beta_m^{i,j},\beta_n^{k,l}] = \delta_{j,k}\,\beta_{m+n}^{i,l} - \delta_{i,l}\,\beta_{m+n}^{k,j}.$$

Defining

$$\tau_m^{i,j} = \mathrm{i}\,(\beta_m^{i,j} - \beta_m^{j,i}),$$

(the $\tau_m^{i,j}$ act as multiplication operators $z^m \otimes T^{i,j}$ on $L^2(S^1)\otimes\mathbb{C}^N$) we obtain a realization of $\widehat{\mathfrak{so}}(N)$ at level zero,

$$[\tau_m^{i,j},\tau_n^{k,l}] = \mathrm{i}\,(\delta_{j,k}\,\tau_{m+n}^{i,l} + \delta_{i,l}\,\tau_{m+n}^{j,k} - \delta_{j,l}\,\tau_{m+n}^{i,k} - \delta_{i,k}\,\tau_{m+n}^{j,k}).$$

Note that skew self-adjoint combinations

$$\mathrm{i}\,\tau_0^{i,j}, \qquad \tau_{m,+}^{i,j} = \mathrm{i}\,(\tau_m^{i,j} + \tau_{-m}^{i,j}), \qquad \tau_{m,-}^{i,j} = \tau_m^{i,j} - \tau_{-m}^{i,j}, \qquad m = 1,2,\dots$$

are elements of $\mathfrak{u}_{P_{\mathrm{NS}}}^{\mathrm{b}}(\mathcal{K},\Gamma)$. Similarly, we define on $\mathcal{K}$ operators λ_m, $m\in\mathbb{Z}$, which act as $-z^m\left(z\frac{\mathrm{d}}{\mathrm{d}z} + \frac{m}{2}\right)$ in each component,

$$\lambda_m = -\sum_{i=1}^{N}\sum_{r\in\mathbb{Z}+1/2} (r + \tfrac{m}{2})|e_{r+m}^i\rangle\langle e_r^i|.$$

Hence

$$[\lambda_m, \lambda_n] = (m-n)\lambda_{m+n},$$

i.e. we obtain a realization of $\mathfrak{Vir}$ with zero central charge (Witt algebra). Note that skew self-adjoint combinations

$$\mathrm{i}\,\lambda_0, \qquad \lambda_{m,+} = \mathrm{i}\,(\lambda_m + \lambda_{-m}), \qquad \lambda_{m,-} = \lambda_m - \lambda_{-m}, \qquad m = 1, 2, \ldots$$

are as in Theorem 2.4. Since also

$$[\lambda_m, \tau_n^{i,j}] = -n\,\tau_{m+n}^{i,j}$$

holds we have together a realization of $\widehat{\mathfrak{so}}(N)_0 \rtimes \mathfrak{Vir}_0$.

3.1.2 Realization of $\widehat{\mathfrak{so}}(N) \rtimes \mathfrak{Vir}$ in the Neveu-Schwarz Sector

We now go on in defining a realization of $\widehat{\mathfrak{so}}(N)_1$ by the procedure of second quantization. Let us denote by $(\mathcal{H}_{\mathrm{NS}}, \pi_{\mathrm{NS}}, |\Omega_{\mathrm{NS}}\rangle)$ the GNS representation of the quasi-free state $\omega_{P_{\mathrm{NS}}}$, and then we define Fourier modes acting on $\mathcal{H}_{\mathrm{NS}}$,

$$b_r^i = \pi_{\mathrm{NS}}(B(e_r^i)), \qquad r \in \mathbb{Z} + \tfrac{1}{2}, \quad i = 1, 2, \ldots, N.$$

Hence we have $(b_r^i)^* = b_{-r}^i$ and anticommutation relations

$$\{b_r^i, b_s^j\} = \delta_{i,j}\,\delta_{r+s,0}\,\mathbf{1},$$

and Fourier modes with positive grade act as annihilation operators in $\mathcal{H}_{\mathrm{NS}}$,

$$b_r^i\,|\Omega_{\mathrm{NS}}\rangle = 0, \quad r > 0.$$

Finite energy vectors

$$b_{-r_m}^{i_m} \cdots b_{-r_2}^{i_2} b_{-r_1}^{i_1} |\Omega_{\mathrm{NS}}\rangle, \qquad r_l \in \mathbb{N}_0 + \tfrac{1}{2}, \qquad i_l = 1, 2, \ldots, N \tag{3.1}$$

are total in $\mathcal{H}_{\mathrm{NS}}$ i.e. finite linear combinations produce a dense subspace $\mathcal{H}_{\mathrm{NS}}^{\mathrm{fin}}$. Denoting normal ordering by colons,

$$:\!b_r^i b_s^j\!: \;\; = \begin{cases} \;\; b_r^i b_s^j & r < 0 \\ -b_s^j b_r^i & r > 0 \end{cases}, \qquad r, s \in \mathbb{Z} + \tfrac{1}{2},$$

we introduce unbounded operators on $\mathcal{H}_{\mathrm{NS}}$

$$B_m^{i,j} = \frac{1}{2} \sum_{r\in\mathbb{Z}+1/2} :b_r^i b_{m-r}^j: \ .$$

Note that these infinite series terminate on finite energy vectors (3.1), i.e. $\mathcal{H}_{\mathrm{NS}}^{\mathrm{fin}}$ is an invariant dense domain of these expressions. For $T \in \mathfrak{so}(N)$ define current operators $J_m(T)$ by

$$J_m(T) = \sum_{i,j=1}^{N} (T)_{i,j} B_m^{i,j}.$$

In particular, $J_m^{i,j} \equiv J_m(T^{i,j})$,

$$J_m^{i,j} = \mathrm{i}\,(B_m^{i,j} - B_m^{j,i}).$$

Then one checks by direct computation

$$[J_m^{i,j}, b_r^k] = \mathrm{i}\,(\delta_{j,k}\, b_{r+m}^i - \delta_{i,k}\, b_{r+m}^j),$$

or equivalently

$$[J_m^{i,j}, \pi_{\mathrm{NS}}(B(f))] = \pi_{\mathrm{NS}}(B(\tau_m^{i,j} f)).$$

Moreover,

$$[J_m^{i,j}, J_n^{k,l}] = \mathrm{i}\,(\delta_{j,k}\, J_{m+n}^{i,l} + \delta_{i,l}\, J_{m+n}^{j,k} - \delta_{j,l}\, J_{m+n}^{i,k} - \delta_{i,k}\, J_{m+n}^{j,l}) + m\,\delta_{m,-n}(T^{i,j}|T^{j,i}), \tag{3.2}$$

i.e. we have a realization of $\widehat{\mathfrak{so}}(N)$ at level 1. It is also straightforward to check that the scalar term on the r.h.s. of Eq. (3.2) is indeed the Schwinger term (2.10),

$$c_{P_{\mathrm{NS}}}(\tau_m^{i,j}, \tau_n^{k,l}) = m\,\delta_{m,-n}(T^{i,j}|T^{k,l}),$$

and hence we identify

$$\mathrm{i}\, J_0^{i,j} = \mathrm{d}Q_{P_{\mathrm{NS}}}(\mathrm{i}\,\tau_0^{i,j}),$$

and

$$\mathrm{i}\,(J_m^{i,j} + J_{-m}^{i,j}) = \mathrm{d}Q_{P_{\mathrm{NS}}}(\tau_{m,+}^{i,j}), \quad J_m^{i,j} - J_{-m}^{i,j} = \mathrm{d}Q_{P_{\mathrm{NS}}}(\tau_{m,-}^{i,j}), \quad m = 1, 2, \ldots$$

Further, we define unbounded operators L_m, $m \in \mathbb{Z}$, on $\mathcal{H}_{\mathrm{NS}}$

$$L_m = -\frac{1}{2} \sum_{i=1}^{N} \sum_{r\in\mathbb{Z}+1/2} (r - \frac{m}{2})\, :b_r^i b_{m-r}^i: \ .$$

Note that also these series terminate on finite energy vectors. One checks by direct computation

$$[L_m, b^i_r] = -(r + \tfrac{m}{2})b^i_{r+m}$$

or, for f in the domain of λ_m,

$$[L_m, \pi_{\mathrm{NS}}(B(f))] = \pi_{\mathrm{NS}}(B(\lambda_m f)).$$

Moreover,

$$[L_m, L_n] = (m-n)L_{m+n} + m(m^2-1)\,\delta_{m,-n}\,\frac{N}{24}\,, \tag{3.3}$$

and

$$[L_m, J^{i,j}_n] = -n\, J^{i,j}_{m+n}, \tag{3.4}$$

i.e. together we have a realization of $\widehat{\mathfrak{so}}(N)_1 \rtimes \mathfrak{Vir}_{N/2}$. Indeed, we identify

$$\mathrm{i}\, L_0 = \mathrm{d}Q_{P_{\mathrm{NS}}}(\mathrm{i}\,\lambda_0),$$

and

$$\mathrm{i}\,(L_m + L_{-m}) = \mathrm{d}Q_{P_{\mathrm{NS}}}(\lambda_{m,+}), \quad L_m - L_{-m} = \mathrm{d}Q_{P_{\mathrm{NS}}}(\lambda_{m,-}), \quad m = 1,2,\ldots$$

3.1.3 Characters

It is known that $\mathcal{H}_{\mathrm{NS}}$ decomposes as a $\widehat{\mathfrak{so}}(N)$ module into the basic module $\mathcal{H}_\circ$ and the vector module $\mathcal{H}_{\mathrm{v}}$ with highest weight vectors $|\Omega_\circ\rangle = |\Omega_{\mathrm{NS}}\rangle$ and $|\Omega_{\mathrm{v}}\rangle = 2^{-1/2}(b^1_{-1/2} + \mathrm{i}b^2_{-1/2})|\Omega_{\mathrm{NS}}\rangle$, respectively. By $\mathbb{Z}_2$-invariance of the current operators this is precisely the decomposition into the even and the odd Fock space, respectively. The corresponding projections $P_\circ \equiv P_+$ and $P_{\mathrm{v}} \equiv P_-$ can be written as

$$P_\pm = \tfrac{1}{2}\,(\mathbf{1} \pm Q_{P_{\mathrm{NS}}}(-\mathbf{1})).$$

It is instructive to compute the Virasoro specialized characters of the irreducible modules. We introduce Euler's product function,

$$\varphi(q) = \prod_{n=1}^{\infty}(1-q^n). \tag{3.5}$$

Thus for the character[1] $\chi_\circ^{(1)}(q) = \mathrm{tr}_{\mathcal{H}_{\mathrm{NS}}}(P_\circ q^{L_0})$ of the basic module we obtain

$$\begin{aligned}\chi_\circ^{(1)}(q) &= \frac{1}{2}\left[\prod_{n=0}^{\infty}(1+q^{n+1/2})^N + \prod_{n=0}^{\infty}(1-q^{n+1/2})^N\right] \\ &= \frac{(\varphi(-q^{1/2}))^N}{2\,(\varphi(q))^N} + \frac{(\varphi(q^{1/2}))^N}{2\,(\varphi(q))^N},\end{aligned}$$

while for the character $\chi_\mathrm{v}^{(1)}(q) = \mathrm{tr}_{\mathcal{H}_{\mathrm{NS}}}(P_\mathrm{v} q^{L_0})$ of the vector module we get

$$\begin{aligned}\chi_\mathrm{v}^{(1)}(q) &= \frac{1}{2}\left[\prod_{n=0}^{\infty}(1+q^{n+1/2})^N - \prod_{n=0}^{\infty}(1-q^{n+1/2})^N\right] \\ &= \frac{(\varphi(-q^{1/2}))^N}{2\,(\varphi(q))^N} - \frac{(\varphi(q^{1/2}))^N}{2\,(\varphi(q))^N}.\end{aligned}$$

3.1.4 Möbius Covariance

Let us give some brief remarks on Möbius covariance of the vacuum sector. Although this is always treated as standard knowledge we have not found a complete proof of Möbius covariance in the literature; therefore we present our own calculations here. The Möbius symmetry on the circle S^1 is given by the group $PSU(1,1) = SU(1,1)/\mathbb{Z}_2$ where

$$SU(1,1) = \left\{ g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in GL(2;\mathbb{C}) \;\middle|\; |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

Its action on the circle is

$$gz = \frac{\overline{\alpha} z - \overline{\beta}}{-\beta z + \alpha}, \qquad z \in S^1.$$

Consider the one-parameter-group of rotations $a_0(t)$,

$$a_0(t) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}t/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}t/2} \end{pmatrix}, \qquad t \in \mathbb{R}.$$

[1] For simplicity, we use the argument $q = \exp(2\pi\mathrm{i}\tau)$, $|q| < 1$, directly instead of the upper complex half plane variable τ. We define the characters simply as the trace of q^{L_0} here, i.e. we also neglect the additional term $-c/24$ in (1.5).

Any element $g \in SU(1,1)$ can be decomposed into a rotation $a_0(t)$ and a transformation $g' = a_0(-t)g$ leaving the point $z = -1$ invariant,

$$g = a_0(t)g', \qquad g' = \begin{pmatrix} \alpha' & \beta' \\ \overline{\beta'} & \overline{\alpha'} \end{pmatrix}, \qquad \frac{\overline{\alpha'} + \overline{\beta'}}{\alpha' + \beta'} = 1.$$

Since $a_0(t+2\pi) = -a_0(t)$ we can determine t, $-2\pi < t \leq 2\pi$ uniquely by the additional requirement $\mathrm{Re}(\alpha') > 0$. Then a representation U of $SU(1,1)$ in our Hilbert space $\mathcal{K}$ of test functions $f = (f^i)_{i=1,2,\ldots,N}$ is defined componentwise by

$$\left(U(g)f\right)^i(z) = \varepsilon(g;z)(\alpha + \overline{\beta}\overline{z})^{-1/2}(\overline{\alpha} + \beta z)^{-1/2} f^i\left(\frac{\alpha z + \overline{\beta}}{\beta z + \overline{\alpha}}\right), \tag{3.6}$$

where for $z = \mathrm{e}^{\mathrm{i}\phi}$, $-\pi < \phi \leq \pi$,

$$\varepsilon(g;z) = -\,\mathrm{sign}(t - \pi - \phi)\ \mathrm{sign}(t + \pi - \phi),$$

and $\mathrm{sign}(x) = 1$ if $x \geq 0$, $\mathrm{sign}(x) = -1$ if $x < 0$. We observe that $\varepsilon(g;z)$ is discontinuous at $z = -1$ and $z = g(-1) = -(\overline{\alpha} + \overline{\beta})(\alpha + \beta)^{-1}$. Up to this ε-factor, Eq. (3.6) is a well-known definition of a representation of $SU(1,1)$. So it remains to be checked that

$$\varepsilon(g_1;z)\varepsilon(g_2;g_1^{-1}z) = \varepsilon(g_1 g_2;z).$$

Since both sides have their discontinuities at $z = -1$ and $z = g_1 g_2(-1)$ they can differ only by a global sign. But this possibility is easily excluded by an argument of L^2-continuity in g. We want to show that U is also unitary. Since the action of $U(g)$ is the same in each component we need only consider the case $N = 1$. Hence we have to establish $\langle U(g)e_r, U(g)e_s\rangle = \delta_{r,s}$ for $r, s \in \mathbb{Z} + \frac{1}{2}$,

$$\begin{aligned}
\langle U(g)e_r, U(g)e_s\rangle &= \oint_{S^1} \frac{\mathrm{d}z}{2\pi \mathrm{i} z}(\alpha + \overline{\beta}\overline{z})^{-1}(\overline{\alpha} + \beta z)^{-1}\left(\frac{\alpha z + \overline{\beta}}{\beta z + \overline{\alpha}}\right)^{s-r} \\
&= \frac{1}{2\pi \mathrm{i}}\oint_{S^1} \mathrm{d}z\ (\alpha z + \overline{\beta})^{s-r-1}(\beta z + \overline{\alpha})^{r-s-1} \\
&= \begin{cases} 0 & s > r \\ \frac{\alpha^{s-r-1}}{(r-s)!}\frac{\mathrm{d}^{r-s}}{\mathrm{d}z^{r-s}}(\beta z + \overline{\alpha})^{r-s-1}\Big|_{z=-\frac{\overline{\beta}}{\alpha}} & s \leq r \end{cases} \\
&= \delta_{r,s}
\end{aligned}$$

by Cauchy's integral formula, respecting that $|\alpha|^2 > |\beta|^2$ since $|\alpha|^2 - |\beta|^2 = 1$. Since the prefactor on the right-hand side in Eq. (3.6) is real we observe $[U(g), \Gamma] = 0$ and hence each $U(g)$, $g \in SU(1,1)$ induces a Bogoliubov automorphism $\alpha_g = \varrho_{U(g)}$ of $\mathcal{C}(\mathcal{K}, \Gamma)$. Hence $SU(1,1)$ is represented by automorphisms of $\mathcal{C}(\mathcal{K}, \Gamma)$, and this restricts to a representation of $PSU(1,1)$ by automorphisms of $\mathcal{C}(\mathcal{K}, \Gamma)^+$. In order to establish Möbius invariance of the vacuum state and hence covariance of the vacuum sector we show that

$$[P_{\mathrm{NS}}, U(g)] = 0, \qquad g \in SU(1,1),$$

i.e. that $U(g)$ respects the polarization of $\mathcal{K}$ induced by P_{NS}. Again we need only consider the case $N = 1$. It is sufficient to show that

$$\langle e_{-r}, U(g) e_s \rangle = 0, \quad r, s \in \mathbb{N}_0 + \tfrac{1}{2}\,, \qquad g \in SU(1,1).$$

The functions $e_r(z)$, $r \in \mathbb{Z} + \frac{1}{2}$ are smooth on S^1 except at their cut at $z = -1$. The prefactor $\varepsilon(g; z)$ in Eq. (3.6) achieves that $(U(g)e_r)(z)$ remains a smooth function except at $z = -1$, i.e. that the cut is not transported to $g(-1)$. Hence we have

$$(U(g)e_r)(z) = \pm(\alpha + \overline{\beta}\overline{z})^{-1/2}(\overline{\alpha} + \beta z)^{-1/2} \left(\frac{\alpha z + \overline{\beta}}{\beta z + \overline{\alpha}} \right)^r ,$$

where all the half-odd integer powers are to be taken in the same branch with cut at $z = -1$. So we can compute as follows:

$$\begin{aligned} \langle e_{-r}, U(g) e_s \rangle &= \pm \oint_{S^1} \frac{\mathrm{d}z}{2\pi \mathrm{i} z} z^r (\alpha z + \overline{\beta})^{-1/2} z^{1/2} (\overline{\alpha} + \beta z)^{-1/2} \left(\frac{\alpha z + \overline{\beta}}{\beta z + \overline{\alpha}} \right)^s \\ &= \pm \frac{1}{2\pi \mathrm{i}} \oint_{S^1} \mathrm{d}z \; z^{r-1/2} (\alpha z + \overline{\beta})^{s-1/2} (\overline{\alpha} + \beta z)^{-s-1/2} = 0, \end{aligned}$$

again by Cauchy's formula, respecting $|\alpha|^2 > |\beta|^2$ and that r, s are positive half-odd integers here.

Consider further one-parameter-subgroups

$$a_+(t) = \begin{pmatrix} \cosh t & \mathrm{i} \sinh t \\ -\mathrm{i} \sinh t & \cosh t \end{pmatrix}, \qquad a_-(t) = \begin{pmatrix} \cosh t & \sinh t \\ -\sinh t & \cosh t \end{pmatrix},$$

$t \in \mathbb{R}$. It is not hard to check that a_0 and $a_\pm$ correspond to infinitesimal generators $\mathrm{i}\,\lambda_0$ and $\lambda_{1,\pm}$, respectively. More precisely,

$$U(a_0(t)) = \exp(\mathrm{i} t \lambda_0), \qquad U(a_\pm(t)) = \exp(t \lambda_{1,\pm})$$

by Stone's Theorem.

3.1.5 Realization of $\widehat{\mathfrak{so}}(N) \rtimes \mathfrak{Vir}$ in the Ramond Sector

We define another (Fourier) orthonormal base

$$\left\{ e_n^i, \quad n \in \mathbb{Z}, \quad i = 1, 2, \dots, N \right\}$$

by the definition

$$e_n^i = e_n \otimes u^i$$

where also $e_n \in L^2(S^1)$ are defined by $e_n(z) = z^n$. Now we define the Ramond operator $S_{\mathrm{R}} \in \mathcal{Q}(\mathcal{K}, \Gamma)$ by

$$S_{\mathrm{R}} = \sum_{i=1}^{N} \left(\frac{1}{2} \, |e_0^i\rangle\langle e_0^i| + \sum_{n \in \mathbb{N}} |e_{-n}^i\rangle\langle e_{-n}^i| \right)$$

and denote by $(\mathcal{H}_{\mathrm{R}}, \pi_{\mathrm{R}}, |\Omega_{\mathrm{R}}\rangle)$ the GNS representation of the associated quasi-free state $\omega_{S_{\mathrm{R}}}$. Further introduce Fourier modes acting on $\mathcal{H}_{\mathrm{R}}$,

$$b_n^i = \pi_{\mathrm{R}}(B(e_n^i)), \qquad n \in \mathbb{Z}, \quad i = 1, 2, \dots, N.$$

Hence we have $(b_n^i)^* = b_{-n}^i$ and anticommutation relations

$$\{b_m^i, b_n^j\} = \delta_{i,j} \, \delta_{m+n,0} \, \mathbf{1},$$

also

$$b_n^i \, |\Omega_{\mathrm{R}}\rangle = 0, \quad n > 0.$$

Finite energy vectors

$$b_{-n_m}^{i_m} \cdots b_{-n_2}^{i_2} b_{-n_1}^{i_1} |\Omega_{\mathrm{R}}\rangle, \qquad n_l \in \mathbb{N}_0, \qquad i_l = 1, 2, \dots, N \tag{3.7}$$

are total in $\mathcal{H}_{\mathrm{R}}$ i.e. finite linear combinations produce a dense subspace $\mathcal{H}_{\mathrm{R}}^{\mathrm{fin}}$. Similar to the situation in $\mathcal{H}_{\mathrm{NS}}$ we denote normal ordering by colons,

$$:b_m^i b_n^j: \;\; = \begin{cases} \;\;\, b_m^i b_n^j & m < 0 \\ -b_n^j b_m^i & m \geq 0 \end{cases}, \qquad m, n \in \mathbb{Z},$$

and introduce unbounded operators on $\mathcal{H}_{\mathrm{R}}$ (by some abuse of notation we employ the same symbols as in the Neveu-Schwarz sector),

$$B_m^{i,j} = \frac{1}{2} \sum_{n \in \mathbb{Z}} :b_n^i b_{m-n}^j: \, .$$

Analogous to the situation in the Neveu-Schwarz sector, $\mathcal{H}_{\mathrm{R}}^{\mathrm{fin}}$ is an invariant dense domain of these expressions. For $T \in \mathfrak{so}(N)$ define current operators $J_m(T)$ by

$$J_m(T) = \sum_{i,j=1}^{N} (T)_{i,j} B_m^{i,j}.$$

In particular, $J_m^{i,j} \equiv J_m(T^{i,j})$,

$$J_m^{i,j} = \mathrm{i}\,(B_m^{i,j} - B_m^{j,i}).$$

We also find by direct computation

$$[J_m^{i,j}, b_n^k] = \mathrm{i}\,(\delta_{j,k}\, b_{n+m}^i - \delta_{i,k}\, b_{n+m}^j)\,.$$

Note that in $\mathcal{H}_{\mathrm{R}}$ the action of $J_m^{i,j}$ comes also from an action $\tau_m^{i,j}$ in $\mathcal{K}$ via $[J_m^{i,j}, \pi_{\mathrm{R}}(B(f))] = \pi_{\mathrm{R}}(B(\tau_m^{i,j} f))$. We have $\tau_m^{i,j} = \mathrm{i}\,(\beta_m^{i,j} - \beta_m^{j,i})$, and one easily verifies that the $J_m^{i,j}$ implement indeed the same action as those in the Neveu-Schwarz sector, i.e.

$$\beta_m^{i,j} = \sum_{n \in \mathbb{Z}} |e_{n+m}^i\rangle\langle e_n^j| = \sum_{r \in \mathbb{Z}+1/2} |e_{r+m}^i\rangle\langle e_r^j|\,.$$

Also (3.2) holds in the Ramond sector, but it is not a direct consequence of Theorem 2.3 since the Ramond state $\omega_{S_{\mathrm{R}}}$ is not pure i.e. not a Fock state. However, there is also a generalization to second quantization in non-Fock states; the Schwinger term (2.10) is just slightly modified in this case, see e.g. [22]. We also define unbounded operators L_m, $m \in \mathbb{Z}$, on $\mathcal{H}_{\mathrm{R}}$ (with invariant dense domain $\mathcal{H}_{\mathrm{R}}^{\mathrm{fin}}$)

$$L_m = -\tfrac{1}{2} \sum_{i=1}^{N} \sum_{n \in \mathbb{Z}} (n - \tfrac{m}{2}\,)\, {:}b_n^i b_{m-n}^i{:} \;+\; \delta_{m,0}\, \frac{N}{16}\; \mathbf{1},$$

Then one checks by direct computation

$$[L_m, b_n^i] = -(n + \tfrac{m}{2}\,) b_{n+m}^i,$$

and also (3.3) and (3.4) hold in the Ramond sector. Note that the L_m in $\mathcal{H}_{\mathrm{R}}$ come also from an action in $\mathcal{K}$ which is $-z^m\left(z\frac{\mathrm{d}}{\mathrm{d}z} + \frac{m}{2}\right)$ in each component, however, these differential operators respect periodic boundary conditions here in contrast to antiperiodic boundary conditions in the Neveu-Schwarz case.

3.1.6 Comparison to the Sectors of the Even CAR Algebra

As a $\widehat{\mathfrak{so}}(N)$ module $\mathcal{H}_{\mathrm{NS}}$ decomposes into the basic ($\circ$) and the vector (v) module. It is also known that $\mathcal{H}_{\mathrm{R}}$ decomposes into the direct sum of 2^ℓ spinor (s) and 2^ℓ conjugate spinor (c) modules if $N = 2\ell$ and into $2^{\ell+1}$ spinor modules (σ) if $N = 2\ell+1$. Using our previous results of CAR theory, we can easily verify that exactly the same happens if we restrict the representations π_{NS} (π_{R}) of $\mathcal{C}(\mathcal{K},\Gamma)$ in $\mathcal{H}_{\mathrm{NS}}$ ($\mathcal{H}_{\mathrm{R}}$) to the even subalgebra $\mathcal{C}(\mathcal{K},\Gamma)^+$: Since P_{NS} is a basis projection we have by Theorem 2.7

$$\pi_{\mathrm{NS}}|_{\mathcal{C}(\mathcal{K},\Gamma)^+} = \pi_{\mathrm{NS}}^+ \oplus \pi_{\mathrm{NS}}^-. \tag{3.8}$$

Now π_{NS}^+ acts in the even Fock space [2] which corresponds to the basic module. Thus we may use the same symbols which label the sectors, $\pi_0 \equiv \pi_{\mathrm{NS}}^+$ (π_0 being the basic, i.e. vacuum representation) and $\pi_{\mathrm{v}} \equiv \pi_{\mathrm{NS}}^-$. Consider the Bogoliubov operator $V_{1/2} \in \mathcal{I}(\mathcal{K},\Gamma)$,

$$V_{1/2} = \sum_{i=1}^{N} \left(|\tilde{e}_+^i\rangle\langle e_0^i| + \sum_{n=1}^{\infty} \left(|e_{n+1/2}^i\rangle\langle e_n^i| + |e_{-n-1/2}^i\rangle\langle e_{-n}^i|\right) \right),$$

where $\tilde{e}_+^i = 2^{-1/2}(e_{1/2}^i + e_{-1/2}^i)$. It is not hard to see that $S_{\mathrm{R}} = V_{1/2}^* P_{\mathrm{NS}} V_{1/2}$, that $M_{V_{1/2}} = N$ and that $N_{V_{1/2}} = 0$. We find $\pi_{\mathrm{R}} \simeq \pi_{\mathrm{NS}} \circ \varrho_{V_{1/2}}$ by Eq. (2.11), and hence by Theorem 2.10,

$$\pi_{\mathrm{R}}|_{\mathcal{C}(\mathcal{K},\Gamma)^+} \simeq \begin{cases} 2^\ell\,(\pi_{P'}^+ \oplus \pi_{P'}^-) & N = 2\ell \\ 2^{\ell+1}\,\pi & N = 2\ell+1 \end{cases} \tag{3.9}$$

for a basis projection P', $[P']_2 = [S_{\mathrm{R}}^{1/2}]_2$. Thus we use notations $\pi_{\mathrm{s}} \equiv \pi_{P'}^+$, $\pi_{\mathrm{c}} \equiv \pi_{P'}^-$ and $\pi_\sigma \equiv \pi$. (Recall that π is one of the equivalent restrictions of the pseudo Fock representations $\pi_{E,\pm}$.) We have seen that the CAR representations π_{NS} and π_{R}, when restricted to the even algebra, reproduce precisely the sectors of the chiral algebra. This is not quite a surprise because the Kac-Moody and Virasoro generators are made of fermion bilinears. Here we see that they indeed act irreducibly in the (dense subspaces of the) sectors of even CAR. This is the reason why we are allowed to identify the elements of the even CAR algebras as the bounded operators representing the observables of the WZW model, and also that we identify the sectors of the even CAR algebra to be the WZW sectors. Note that the Bogoliubov endomorphism $\varrho_{V_{1/2}}$ induces a transition from the vacuum sector to spinor sectors.

3.2 Treatment in the Algebraic Framework

Our incorporation of the level 1 $\mathfrak{so}(N)$ WZW models in the framework of AQFT is based on the fact that (local) even CAR algebras can be identified as (local) observable algebras. We proceed as follows: We introduce a system of local even CAR algebras on the circle. Then we can define localized endomorphisms in terms of Bogoliubov transformations. Later we extend representations and endomorphisms to a net of von Neumann algebras on the punctured circle, and this will be the foundation for the proof of the fusion rules using the DHR sector product.

3.2.1 Localized Endomorphisms

We introduce at first a local structure on S^1, i.e. we define local algebras of observables. Let us denote by $\mathcal{J}$ the set of open, non-void proper subintervals of S^1. For $I \in \mathcal{J}$ set $\mathcal{K}(I) = L^2(I; \mathbb{C}^N)$ and define local C^*-algebras

$$\mathfrak{C}(I) = \mathcal{C}(\mathcal{K}(I), \Gamma)^+$$

so that we have inclusions

$$\mathfrak{C}(I_1) \subset \mathfrak{C}(I_0), \qquad I_1 \subset I_0,$$

inherited by the natural embedding of the L^2-spaces; and also we have locality,

$$[\mathfrak{C}(I_1), \mathfrak{C}(I_2)] = \{0\}, \qquad I_1 \cap I_2 = \emptyset.$$

Our construction of localized endomorphisms happens on the punctured circle. Consider the interval $I_\zeta \in \mathcal{J}$ which is S^1 by removing one "point at infinity" $\zeta \in S^1$, $I_\zeta = S^1 \setminus \{\zeta\}$. Clearly, $\mathfrak{C}(I_\zeta) = \mathcal{C}(\mathcal{K}, \Gamma)^+$. Further denote by $\mathcal{J}_\zeta$ the set of "finite" intervals $I \in \mathcal{J}$ such that their closure is contained in I_ζ,

$$\mathcal{J}_\zeta = \{ I \in \mathcal{J} \mid \bar{I} \subset I_\zeta \}.$$

An endomorphism ϱ of $\mathfrak{C}(I_\zeta)$ is called localized in some interval $I \in \mathcal{J}_\zeta$ if it satisfies

$$\varrho(A) = A, \qquad A \in \mathfrak{C}(I_1), \qquad I_1 \in \mathcal{J}_\zeta, \qquad I_1 \cap I = \emptyset.$$

The construction of localized endomorphisms by means of Bogoliubov transformations leads to the concept of pseudo-localized isometries [41]. For

$I \in \mathcal{J}_\zeta$ denote by I_+ and I_- the two connected components of $I' \cap I_\zeta$ (I' always denotes the interior of the complement of I in S^1, $I' = I^{\mathrm{c}} \setminus \partial I^{\mathrm{c}}$). A Bogoliubov operator $V \in \mathcal{I}(\mathcal{K}, \Gamma)$ is called even (resp. odd) pseudo-localized in $I \in \mathcal{J}_\zeta$ if

$$V f = \epsilon_\pm f, \qquad f \in \mathcal{K}(I_\pm), \qquad \epsilon_\pm \in \{-1, 1\},$$

and $\epsilon_+ = \epsilon_-$ (resp. $\epsilon_+ = -\epsilon_-$). Then, as obvious, ϱ_V is localized in I in restriction to $\mathfrak{C}(I_\zeta)$. Now we are ready to define our localized vector endomorphism.

Definition 3.1 *For some $I \in \mathcal{J}_\zeta$ choose a real $v \in \mathcal{K}(I)$, $\Gamma v = v$ and $\|v\| = 1$. Define the unitary self-adjoint Bogoliubov operator $U \in \mathcal{I}(\mathcal{K}, \Gamma)$ by*

$$U = 2|v\rangle\langle v| - \mathbf{1}, \tag{3.10}$$

and the localized vector endomorphism (automorphism) ϱ_{v} by $\varrho_{\mathrm{v}} = \varrho_U$.

Since U is even pseudo-localized, and by Corollary 2.8, ϱ_{v} is indeed a localized vector endomorphism, i.e. $\pi_0 \circ \varrho_{\mathrm{v}} \simeq \pi_{\mathrm{v}}$. Further, by $U^2 = \mathbf{1}$ we have $\pi_0 \circ \varrho_{\mathrm{v}}^2 \simeq \pi_0$. It follows also from Corollary 2.8 that $\pi_{\mathrm{s}} \circ \varrho_{\mathrm{v}} \simeq \pi_{\mathrm{c}}$. The construction of a localized spinor endomorphism is a little bit more costly. Without loss of generality, we choose $\zeta = -1$ and the localization region to be I_2,

$$I_2 = \left\{ z = \mathrm{e}^{\mathrm{i}\phi} \in S^1 \;\middle|\; -\tfrac{\pi}{2} < \phi < \tfrac{\pi}{2} \right\}$$

such that the connected components $I_\pm$ of $I_2' \cap I_\zeta$ are given by

$$\begin{aligned} I_- &= \left\{ z = \mathrm{e}^{\mathrm{i}\phi} \in S^1 \;\middle|\; -\pi < \phi < -\tfrac{\pi}{2} \right\}, \\ I_+ &= \left\{ z = \mathrm{e}^{\mathrm{i}\phi} \in S^1 \;\middle|\; \tfrac{\pi}{2} < \phi < \pi \right\}. \end{aligned}$$

Our Hilbert space $\mathcal{K} = \mathcal{K}(I_\zeta)$ decomposes into a direct sum,

$$\mathcal{K} = \mathcal{K}(I_-) \oplus \mathcal{K}(I_2) \oplus \mathcal{K}(I_+).$$

By P_{I_+}, P_{I_-} we denote the projections onto the subspaces $\mathcal{K}(I_+)$, $\mathcal{K}(I_-)$, respectively. Define functions on S^1 by

$$f_p(z) = \begin{cases} \sqrt{2}\, z^{2p} & z \in I_2 \\ 0 & z \notin I_2 \end{cases}, \qquad p \in \tfrac{1}{2}\mathbb{Z},$$

and

$$f_p^i = f_p \otimes u^i, \qquad p \in \tfrac{1}{2}\,\mathbb{Z}, \qquad i = 1, 2, \dots, N,$$

such that we obtain two ONB of the subspace $\mathcal{K}(I_2) \subset \mathcal{K}$,

$$\{f_r^i, r \in \mathbb{Z} + \tfrac{1}{2}\,, i = 1, 2, \dots, N\}, \qquad \{f_n^i, n \in \mathbb{Z}, i = 1, 2, \dots, N\}.$$

Now define the odd pseudo-localized Bogoliubov operator $V \in \mathcal{I}(\mathcal{K}, \Gamma)$,

$$\begin{aligned}
V &= P_{I_-} - P_{I_+} + V^{(2)}, \\
V^{(2)} &= \sum_{\substack{j \le N \\ j \text{ odd}}} (\mathrm{i}r^j + \mathrm{i}R^j) - \sum_{\substack{j \le N \\ j \text{ even}}} (t^j + \mathrm{i}T^j), \\
r^j &= \frac{1}{\sqrt{2}}\,|f_{1/2}^j\rangle\langle f_0^j| - \frac{1}{\sqrt{2}}\,|f_{-1/2}^j\rangle\langle f_0^j|, \\
R^j &= \sum_{n=1}^{\infty} \Big(|f_{n+1/2}^j\rangle\langle f_n^j| - |f_{-n-1/2}^j\rangle\langle f_{-n}^j| \Big), \\
t^j &= \frac{1}{\sqrt{2}}\,|f_{1/2}^{j-1}\rangle\langle f_0^j| + \frac{1}{\sqrt{2}}\,|f_{-1/2}^{j-1}\rangle\langle f_0^j|, \\
T^j &= \sum_{n=1}^{\infty} \Big(|f_{n-1/2}^j\rangle\langle f_n^j| - |f_{-n+1/2}^j\rangle\langle f_{-n}^j| \Big).
\end{aligned}$$

Note that V is unitary if $N = 2\ell$. More precisely, we have

$$M_V = \begin{cases} 0 & N = 2\ell \\ 1 & N = 2\ell + 1 \end{cases}.$$

Furthermore, we claim

Lemma 3.2 *With notations as above,*

$$[(V^* P_{\mathrm{NS}} V)^{1/2}]_2 = [S_{\mathrm{R}}^{1/2}]_2, \tag{3.11}$$

$$[(V^* V^* P_{\mathrm{NS}} V V)^{1/2}]_2 = [P_{\mathrm{NS}}]_2. \tag{3.12}$$

Proof. Let us first point out that that we do not have to take care about the positive square roots because for any basis projection P and any Bogoliubov operator $W \in \mathcal{I}(\mathcal{K}, \Gamma)$ with $M_W < \infty$ we have

$$[(W^* P W)^{1/2}]_2 = [W^* P W]_2$$

since

$$\begin{aligned}\|(W^*PW)^{1/2} - W^*PW\|_2^2 &\leq \|W^*PW - (W^*PW)^2\|_1 \\ &= \|W^*P(\mathbf{1} - WW^*)PW\|_1 \\ &\leq \|W\|^2\|P\|^2\|\mathbf{1} - WW^*\|_1 = M_W.\end{aligned}$$

We used the trace norm and Hilbert Schmidt norm $\|A\|_n = (\mathrm{tr}(A^*A)^{n/2})^{1/n}$, $n = 1, 2$, respectively, and also an estimate [45]

$$\|A^{1/2} - B^{1/2}\|_2^2 \leq \|A - B\|_1, \qquad A, B \in \mathfrak{B}(\mathcal{K}), \qquad A, B \geq 0. \tag{3.13}$$

It was proven in [5], Lemma 3.10, that

$$V^*P_{\mathrm{NS}}V - S_{\mathrm{R}}, \quad VP_{\mathrm{NS}}V^* - S_{\mathrm{R}}, \quad V'^*P_{\mathrm{NS}}V' - S_{\mathrm{R}}, \quad V'P_{\mathrm{NS}}V'^* - S_{\mathrm{R}}$$

are Hilbert Schmidt operators for the case $N = 1$, where in our notation

$$V = P_{I_-} - P_{I_+} + \mathrm{i}r^1 + \mathrm{i}R^1, \qquad V' = P_{I_-} - P_{I_+} + \mathrm{i}(T^1)^*.$$

For arbitrary N, operators $V^*P_{\mathrm{NS}}V - S_{\mathrm{R}}$ and $VP_{\mathrm{NS}}V^* - S_{\mathrm{R}}$ are just direct sums of the above Hilbert Schmidt operators (up to finite dimensional operators), hence we conclude for arbitrary N

$$V^*P_{\mathrm{NS}}V - S_{\mathrm{R}} \in \mathfrak{J}_2(\mathcal{K}), \qquad VP_{\mathrm{NS}}V^* - S_{\mathrm{R}} \in \mathfrak{J}_2(\mathcal{K}).$$

But both relations together imply that $P_{\mathrm{NS}} - V^*V^*P_{\mathrm{NS}}VV$ is also Hilbert Schmidt, and this proves the lemma. □

Hence we conclude $\pi_{\mathrm{NS}} \circ \varrho_V \approx \pi_{\mathrm{R}}$. For $N = 2\ell$ the basis projection $P' = V^*P_{\mathrm{NS}}V$ is as in Eq. (3.9). For $N = 2\ell + 1$ the representation $\pi_{\mathrm{NS}} \circ \varrho_V$, when restricted to $\mathcal{C}(\mathcal{K}, \Gamma)^+$, decomposes into two equivalent irreducibles. With our above definitions and using Corollary 2.8, this suggests the following

Definition 3.3 *Choose $U \in \mathcal{I}(\mathcal{K}, \Gamma)$ for $v \in \mathcal{K}(I_2)$ as in Definition 3.1. For $N = 2\ell$ define the localized spinor endomorphism ϱ_{s} by $\varrho_{\mathrm{s}} = \varrho_V$ and the localized conjugate spinor endomorphism ϱ_{c} by $\varrho_{\mathrm{c}} = \varrho_U\varrho_V$. For $N = 2\ell + 1$ define the localized spinor endomorphism ϱ_σ by $\varrho_\sigma = \varrho_V$.*

Note that this definition fixes the choice, if N is even, which of the two inequivalent spinor sectors is called s and which c. This might seem to be somewhat inconsistent because for the highest weight modules there is no ambiguity within $\mathcal{H}_{\mathrm{R}}$, for instance, $2^{-\ell}\prod_{j=1}^{\ell}(\mathbf{1}-2\,\mathrm{i}\,b_0^{2j}b_0^{2j-1})|\Omega_{\mathrm{R}}\rangle$ is a highest weight vector of weight Λ_{s}. However, for the sectors of even CAR we take the freedom to rename the sectors i.e. which of the spinor sectors is called s and which c. There is no problem with the proof of the fusion rules later on since they are invariant under simultaneous exchange of s and c. Indeed, our considerations have shown

Theorem 3.4 *The localized endomorphisms of Definitions 3.1 and 3.3 satisfy* $\pi_0 \circ \varrho_\xi \simeq \pi_\xi$, $\xi = \mathrm{v}, \mathrm{s}, \mathrm{c}, \sigma$.

3.2.2 Extension to Local von Neumann Algebras

We have obtained the relevant localized endomorphisms which generate the sectors $\mathrm{v}, \mathrm{s}, \mathrm{c}, \sigma$. It is our next aim to derive fusion rules in terms of DHR sectors i.e. of unitary equivalence classes $[\pi_0 \circ \varrho]$ for localized endomorphisms ϱ. For such a formulation one needs local intertwiners in the observable algebra. So we have to keep close to the DHR framework, in particular, we should use local von Neumann algebras instead of local C^*-algebras $\mathfrak{C}(I)$. We define

$$\mathcal{R}(I) = \pi_0(\mathfrak{C}(I))'', \qquad I \in \mathcal{J}.$$

By Möbius covariance of the vacuum state, this defines a so-called covariant precosheaf on the circle [9]. In particular, we have Haag duality,

$$\mathcal{R}(I)' = \mathcal{R}(I'). \tag{3.14}$$

Since the set $\mathcal{J}$ is not directed by inclusion we cannot define a global algebra as the C^*-norm closure of the union of all local algebras. However, the set $\mathcal{J}_\zeta$ is directed so that we can define the following algebra $\mathcal{A}$ of quasilocal observables in the usual manner,

$$\mathcal{A} = \overline{\bigcup_{I \in \mathcal{J}_\zeta} \mathcal{R}(I)}.$$

We want to prove that Haag duality holds also on the punctured circle and need some technical preparation. Recall that a function $k \in L^2(S^1)$ is in the Hardy space H^2 if $\langle e_{-n}, k\rangle = 0$ for all $n \in \mathbb{N}$ where $e_{-n}(z) = z^{-n}$.

There is a Theorem of Riesz [21, Th. 6.13] which states that $k(z) \neq 0$ almost everywhere if $k \in H^2$ is non-zero. Now suppose $f \in P_{\mathrm{NS}}\mathcal{K}$. Then $g^i \in H^2$ where $g^i(z) = z^{1/2}\overline{f^i(z)}$ component-wise, $i = 1, 2, \ldots, N$. We conclude

Lemma 3.5 *If $f \in P_{\mathrm{NS}}\mathcal{K}$ then $f \in \mathcal{K}(I)$ implies $f = 0$ for any $I \in \mathcal{J}$.*

For some interval $I \in \mathcal{J}_\zeta$, let us denote by $\mathcal{A}_\zeta(I')$ the norm closure of the algebra generated by all $\mathcal{R}(I_1)$, $I_1 \in \mathcal{J}_\zeta$, $I_1 \cap I = \emptyset$. Obviously $\mathcal{A}_\zeta(I')'' \subset \mathcal{R}(I')$; a key point of the analysis is the following

Lemma 3.6 *Haag duality remains valid on the punctured circle, i.e.*

$$\mathcal{R}(I)' = \mathcal{A}_\zeta(I')''. \tag{3.15}$$

Proof. We have to prove $\mathcal{A}_\zeta(I')'' = \mathcal{R}(I')$. It is sufficient to show that each generator $\pi_0(B(f)B(g))$, $f, g \in \mathcal{K}(I')$ of $\mathcal{R}(I')$ is a weak limit point of a net in $\mathcal{A}_\zeta(I')$. Note that the subspace $\mathcal{K}^{(\zeta)}(I') \subset \mathcal{K}(I')$ of functions which vanish in a neighborhood of ζ is dense. So by Eq. (2.1) we conclude that it is sufficient to establish this fact only for such generators with $f, g \in \mathcal{K}^{(\zeta)}(I')$, because these generators approximate the arbitrary ones already in the norm topology. Let us again denote the two connected components of $I' \setminus \{\zeta\}$ by I_+ and I_-, and the projections onto corresponding subspaces $\mathcal{K}(I_\pm)$ by $P_\pm$. We also write $f_\pm = P_\pm f$ and $g_\pm = P_\pm g$ for our functions $f, g \in \mathcal{K}^{(\zeta)}(I')$. Then we have

$$\begin{aligned} \pi_0(B(f)B(g)) &= \pi_0(B(f_+)B(g_+)) + \pi_0(B(f_-)B(g_-)) \\ &\quad + \pi_0(B(f_+)B(g_-)) + \pi_0(B(f_-)B(g_+)). \end{aligned}$$

Clearly, the first two terms on the r.h.s. are elements of $\mathcal{A}_\zeta(I')$. We show that the third term $Y = \pi_0(B(f_+)B(g_-))$ (then, by symmetry, also the fourth one) is in $\mathcal{A}_\zeta(I')''$. In the same way as in the proof of Lemma 4.1 in [5] one constructs a sequence $\{X_n, n \in \mathbb{N}\}$,

$$X_n = \pi_0(B(h_n^+)B(h_n^-))$$

where unit vectors $h_n^\pm \in \mathcal{K}(I_n^\pm)$ are related by Möbius transformations such that intervals $I_n^\pm \subset I_\pm$ shrink to the point ζ. Since $\|X_n\| \leq 1$ by Eq. (2.1) it follows that there is a weakly convergent subnet $\{Z_\alpha, \alpha \in \iota\}$ (ι a directed set), w-$\lim_\alpha Z_\alpha = Z$. For each $I_0 \in \mathcal{J}_\zeta$ elements X_n commute with each $A \in \mathcal{R}(I_0)$ for sufficiently large n. Hence Z is in the commutant of $\mathcal{A}$ and

this implies $Z = \lambda \mathbf{1}$. We have chosen the vectors $h_n^\pm$ related by Möbius transformations. By Möbius invariance of the vacuum state we have

$$\lambda = \langle \Omega_0 | X_1 | \Omega_0 \rangle = \langle \Gamma h_1^+, P_{\mathrm{NS}} h_1^- \rangle.$$

We claim that we can choose $h_1^\pm$ such that $\lambda \neq 0$. For given h_1^- set $k = P_{\mathrm{NS}} h_1^-$. We have $k \neq 0$, otherwise $\Gamma h_1^- \in P_{\mathrm{NS}} \mathcal{K}$ in contradiction to $h_1^- \in \mathcal{K}(I_1^-)$ by Lemma 3.5. Again by Lemma 3.5 we conclude that k cannot vanish almost everywhere. So we clearly can choose a function $h_1^+ \in \mathcal{K}(I_1^+)$ such that $\lambda = \langle \Gamma h_1^+, k \rangle \neq 0$. Now we find $Y = \lambda^{-1} \text{w-lim}_\alpha Y Z_\alpha$ and also $Y Z_\alpha \in \mathcal{A}_\zeta(I')$ because

$$\begin{aligned} Y X_n &= \pi_0(B(f_+) B(g_-) B(h_n^+) B(h_n^-)) \\ &= -\pi_0(B(f_+) B(h_n^+)) \pi_0(B(g_-) B(h_n^-)) \end{aligned}$$

is in $\mathcal{A}_\zeta(I')$ for all $n \in \mathbb{N}$. □

Since the vacuum representation is faithful on $\mathfrak{C}(I_\zeta)$ we can identify observables A in the usual manner with their vacuum representers $\pi_0(A)$. Thus we consider the vacuum representation as acting as the identity on $\mathcal{A}$, and, in the same fashion, we treat local C^*-algebras as subalgebras $\mathfrak{C}(I) \subset \mathcal{R}(I)$. Now we have to check whether we can extend our representations π_ξ and endomorphisms ϱ_ξ from $\mathfrak{C}(I)$ to $\mathcal{R}(I) = \mathfrak{C}(I)''$, $I \in \mathcal{J}_\zeta$, $\xi = \mathrm{v}, \mathrm{s}, \mathrm{c}, \sigma$. That is that we have to check local quasi-equivalence of the representations π_ξ and this will now be elaborated. Define $E_{\mathrm{R}} \in \mathfrak{B}(\mathcal{K})$ by

$$E_{\mathrm{R}} = \sum_{i=1}^{N} \sum_{n \in \mathbb{N}} |e_{-n}^i\rangle\langle e_{-n}^i| + \sum_{\substack{j \leq N \\ j \text{ even}}} |e_+^j\rangle\langle e_+^j|$$

where $e_+^j = 2^{-1/2}(e_0^j + \mathrm{i} e_0^{j-1})$.

Lemma 3.7 *For $I \in \mathcal{J}$ the subspaces $P_{\mathrm{NS}}\mathcal{K}(I) \subset P_{\mathrm{NS}}\mathcal{K}$ and $E_{\mathrm{R}}\mathcal{K}(I) \subset E_{\mathrm{R}}\mathcal{K}$ are dense.*

Proof. Suppose that $P_{\mathrm{NS}}\mathcal{K}(I)$ is not dense in $P_{\mathrm{NS}}\mathcal{K}$. Then there is a non-zero $f \in P_{\mathrm{NS}}\mathcal{K}$ such that

$$\langle f, P_{\mathrm{NS}} g \rangle = \langle f, g \rangle = 0$$

for all $g \in \mathcal{K}(I)$. Hence $f \in \mathcal{K}(I)^\perp = \mathcal{K}(I')$ in contradiction to Lemma 3.5. As quite obvious, Lemma 3.5 holds for $f \in E_{\mathrm{R}}\mathcal{K}$ as well. So also $E_{\mathrm{R}}\mathcal{K}(I)$ is dense in $E_{\mathrm{R}}\mathcal{K}$. □

Note that E_{R} is a basis projection if N is even. For N odd, E_{R} is a partial basis projection with Γ-codimension 1 and corresponding Γ-invariant unit vector e_0^N. In this case

$$S'_{\mathrm{R}} = \frac{1}{2}\,|e_0^N\rangle\langle e_0^N| + E_{\mathrm{R}}$$

is of the form (2.4). Let us denote by $(\mathcal{H}_{\mathrm{R}'}, \pi_{\mathrm{R}'}, |\Omega_{\mathrm{R}'}\rangle)$ the GNS representation of the quasi-free state $\omega_{E_{\mathrm{R}}}$ if N is even and $\omega_{S'_{\mathrm{R}}}$ if N is odd. We conclude

$$\pi_{\mathrm{R}'}|_{\mathcal{C}(\mathcal{K},\Gamma)^+} \simeq \begin{cases} \pi_{\mathrm{s}} \oplus \pi_{\mathrm{c}} & N = 2\ell \\ 2\pi_\sigma & N = 2\ell+1 \end{cases}$$

by Theorem 2.7 and Lemma 2.1 and the fact that $[E_{\mathrm{R}}]_2 = [S_{\mathrm{R}}^{1/2}]_2 = [S_{\mathrm{R}}]_2$ (N even) and $[{S'_{\mathrm{R}}}^{1/2}]_2 = [S'_{\mathrm{R}}]_2 = [S_{\mathrm{R}}]_2$ (N odd).

Lemma 3.8 *For $I \in \mathcal{J}_\zeta$ we have local quasi-equivalence*

$$\pi_{\mathrm{NS}}|_{\mathcal{C}(\mathcal{K}(I),\Gamma)} \approx \pi_{\mathrm{R}'}|_{\mathcal{C}(\mathcal{K}(I),\Gamma)}. \tag{3.16}$$

Proof. We first claim that $|\Omega_{\mathrm{NS}}\rangle$ and $|\Omega_{\mathrm{R}'}\rangle$ remain cyclic for $\pi_{\mathrm{NS}}(\mathcal{C}(\mathcal{K}(I),\Gamma))$ and $\pi_{\mathrm{R}'}(\mathcal{C}(\mathcal{K}(I),\Gamma))$, respectively. By Lemma 3.7, $P_{\mathrm{NS}}\mathcal{K}(I) \subset P_{\mathrm{NS}}\mathcal{K}$ is dense. It follows that vectors of the form $\pi_{\mathrm{NS}}(B(f_1)\cdots B(f_n))|\Omega_{\mathrm{NS}}\rangle$, with $f_1, f_2, \ldots, f_n \in P_{\mathrm{NS}}\mathcal{K}(I)$, $n = 0, 1, 2, \ldots$, are total in $\mathcal{H}_{\mathrm{NS}}$. This proves the required cyclicity of $|\Omega_{\mathrm{NS}}\rangle$. For N even, cyclicity of $|\Omega_{\mathrm{R}'}\rangle$ for $\mathcal{C}(\mathcal{K}(I),\Gamma)$ is proven in the same way. For N odd, we have $\mathcal{H}_{\mathrm{R}'} = \mathcal{H}_{E_{\mathrm{R}}} \oplus \mathcal{H}_{E_{\mathrm{R}}}$, $\pi_{\mathrm{R}'} = \pi_{E,+} \oplus \pi_{E,-}$ and $|\Omega_{\mathrm{R}'}\rangle = 2^{-1/2}(|\Omega_{E_{\mathrm{R}}}\rangle \oplus |\Omega_{E_{\mathrm{R}}}\rangle)$ as in Lemma 2.1, and the corresponding Γ-invariant unit vector is given by e_0^N. In order to prove cyclicity of $|\Omega_{\mathrm{R}'}\rangle$ we show that $\langle\Psi|\pi_{\mathrm{R}'}(x)|\Omega_{\mathrm{R}'}\rangle = 0$ for all $x \in \mathcal{C}(\mathcal{K}(I),\Gamma)$, $|\Psi\rangle = |\Psi_+\rangle \oplus |\Psi_-\rangle \in \mathcal{H}_{\mathrm{R}'}$, implies $|\Psi\rangle = 0$. We have

$$\langle\Psi|\pi_{\mathrm{R}'}(x)|\Omega_{\mathrm{R}'}\rangle = \frac{1}{\sqrt{2}}\,\langle\Psi_+|\pi_{E_{\mathrm{R}},+}(x)|\Omega_{E_{\mathrm{R}}}\rangle + \frac{1}{\sqrt{2}}\,\langle\Psi_-|\pi_{E_{\mathrm{R}},-}(x)|\Omega_{E_{\mathrm{R}}}\rangle = 0$$

Again by Lemma 3.7, $E_{\mathrm{R}}\mathcal{K}(I) \subset E_{\mathrm{R}}\mathcal{K}$ is dense, hence vectors of the form $\pi_{E_{\mathrm{R}},\pm}(x)|\Omega_{E_{\mathrm{R}}}\rangle = \pi_{E_{\mathrm{R}}}(x)|\Omega_{E_{\mathrm{R}}}\rangle$, $x = B(f_1)\cdots B(f_n)$, $f_1, f_2, \ldots, f_n \in E_{\mathrm{R}}\mathcal{K}(I)$, $n = 0, 1, 2, \ldots$, are total in $\mathcal{H}_{E_{\mathrm{R}}}$. It follows $|\Psi_-\rangle = -|\Psi_+\rangle$. Hence

$$\langle\Psi_+|(\pi_{E_{\mathrm{R}},+}(y) - \pi_{E_{\mathrm{R}},-}(y))|\Omega_{E_{\mathrm{R}}}\rangle = 0, \qquad y \in \mathcal{C}(\mathcal{K}(I),\Gamma).$$

Keep all $x = B(f_1)\cdots B(f_n)$ as above and choose an $f \in \mathcal{K}(I)$ such that $\langle e_0^N, f\rangle = 2^{-1/2}$. Set $y = (-1)^n B(f)x$. Then, by Eq. (2.3), we compute

$$\pi_{E_{\mathrm{R}},\pm}(y) = (-1)^n \left(\pm\frac{1}{2}\, Q_{E_{\mathrm{R}}}(-\mathbf{1}) + \pi_{E_{\mathrm{R}}}(B((E_{\mathrm{R}} + \overline{E}_{\mathrm{R}})f))\right) \pi_{E_{\mathrm{R}}}(x)$$

and hence

$$(\pi_{E_{\mathrm{R}},+}(y) - \pi_{E_{\mathrm{R}},-}(y))|\Omega_{E_{\mathrm{R}}}\rangle = \pi_{E_{\mathrm{R}}}(B(f_1)\cdots B(f_n))|\Omega_{E_{\mathrm{R}}}\rangle.$$

Because such vectors are total in $\mathcal{H}_{E_{\mathrm{R}}}$ we find $|\Psi_+\rangle = 0$ and hence $|\Psi\rangle = 0$. We have seen that vectors $|\Omega_{\mathrm{NS}}\rangle$ and $|\Omega_{\mathrm{R}'}\rangle$ remain cyclic. Thus we can prove the lemma by showing that the restricted states $\omega_{P_I P_{\mathrm{NS}} P_I}$ and $\omega_{P_I E_{\mathrm{R}} P_I}$ ($N = 2\ell$) respectively $\omega_{P_I S'_{\mathrm{R}} P_I}$ ($N = 2\ell + 1$) give rise to quasi-equivalent representations. Because they are quasi-free on $\mathcal{C}(\mathcal{K}(I), \Gamma)$ we have to show that

$$[(P_I P_{\mathrm{NS}} P_I)^{1/2}]_2 = \begin{cases} [(P_I E_{\mathrm{R}} P_I)^{1/2}]_2 & N = 2\ell \\ [(P_I S'_{\mathrm{R}} P_I)^{1/2}]_2 & N = 2\ell + 1 \end{cases}.$$

By use of Eq. (3.13) it is sufficient to show that the difference of $P_I P_{\mathrm{NS}} P_I$ and $P_I E_{\mathrm{R}} P_I$ respectively $P_I S'_{\mathrm{R}} P_I$ is trace class for $I \in \mathcal{J}_\varsigma$. It is obviously sufficient to prove that $P_I P_{\mathrm{NS}} P_I - P_I S_{\mathrm{R}} P_I$ is trace class for the case $N = 1$ (since all the operators above are, up to finite dimensional operators, direct sums of those for $N = 1$). We use the parameterization $z = \mathrm{e}^{\mathrm{i}\phi}$, $-\pi < \phi \leq \pi$ of S^1. Recall that Hilbert Schmidt operators $A \in \mathcal{J}_2(L^2(S^1))$ can be written as square integrable kernels $A(\phi, \phi')$. For instance, a rank-one-projection $|e_r\rangle\langle e_r|$ has kernel $\mathrm{e}^{\mathrm{i}r(\phi-\phi')}$. For (small) $\epsilon > 0$ define operators in $P_{\mathrm{NS}}^{(\epsilon)}, S_{\mathrm{R}}^{(\epsilon)} \in \mathcal{J}_2(L^2(S^1))$ by kernels

$$P_{\mathrm{NS}}^{(\epsilon)}(\phi, \phi') = \sum_{n=0}^{\infty} \mathrm{e}^{-(n+1/2)(\mathrm{i}\phi - \mathrm{i}\phi' + \epsilon)} = \frac{\mathrm{e}^{-(\mathrm{i}\phi - \mathrm{i}\phi' + \epsilon)/2}}{1 - \mathrm{e}^{-(\mathrm{i}\phi - \mathrm{i}\phi' + \epsilon)}},$$

and

$$S_{\mathrm{R}}^{(\epsilon)}(\phi, \phi') = \frac{1}{2} + \sum_{n=1}^{\infty} \mathrm{e}^{-n(\mathrm{i}\phi - \mathrm{i}\phi' + \epsilon)} = \frac{1}{1 - \mathrm{e}^{-(\mathrm{i}\phi - \mathrm{i}\phi' + \epsilon)}} - \frac{1}{2}.$$

Note that $\epsilon > 0$ regularizes the singularities for $\phi - \phi' = 0, \pm 2\pi$. Using Cauchy's integral formula, it is easy to check that for $r, s \in \mathbb{Z} + \frac{1}{2}$,

$$\lim_{\epsilon \searrow 0} \langle e_r, P_{\mathrm{NS}}^{(\epsilon)} e_s\rangle \quad = \quad \lim_{\epsilon \searrow 0} \oint_{S^1} \frac{\mathrm{d}z}{2\pi\mathrm{i}z} \oint_{S^1} \frac{\mathrm{d}z'}{2\pi\mathrm{i}z'}\, \frac{z^{-r+1/2} z'^{s+1/2} \mathrm{e}^{-\epsilon/2}}{z - z'\mathrm{e}^{-\epsilon}}$$

$$
\begin{aligned}
&= \begin{cases} \lim_{\epsilon\searrow 0}\ \mathrm{e}^{s\epsilon}\,\delta_{r-s,0} & s<0 \\ 0 & \text{otherwise} \end{cases} \\
&= \langle e_r, P_{\mathrm{NS}} e_s\rangle .
\end{aligned}
$$

Because $\mathrm{e}^{s\epsilon} < 1$ for $s<0$ this result can be generalized to

$$\lim_{\epsilon\searrow 0} \langle f, P_{\mathrm{NS}}^{(\epsilon)} g\rangle = \langle f, P_{\mathrm{NS}} g\rangle$$

for arbitrary $f, g \in L^2(S^1)$ by an argument of bounded convergence. So we have weak convergence $\text{w-}\lim_{\epsilon\searrow 0} P_{\mathrm{NS}}^{(\epsilon)} = P_{\mathrm{NS}}^{(0)} \equiv P_{\mathrm{NS}}$. In an analogous way one obtains $\text{w-}\lim_{\epsilon\searrow 0} S_{\mathrm{R}}^{(\epsilon)} = S_{\mathrm{R}}^{(0)} \equiv S_{\mathrm{R}}$. Thus the difference $\Delta^{(\epsilon)} = S_{\mathrm{R}}^{(\epsilon)} - P_{\mathrm{NS}}^{(\epsilon)}$ with kernel

$$\Delta^{(\epsilon)}(\phi,\phi') = \frac{1}{1+\mathrm{e}^{-(\mathrm{i}\phi-\mathrm{i}\phi'+\epsilon)/2}} - \frac{1}{2}$$

converges weakly to $\Delta = S_{\mathrm{R}} - P_{\mathrm{NS}}$. We have to show that $X = P_I \Delta P_I$ is trace class. The operator P_I acts as multiplication with the characteristic function $\chi_I(\phi)$ corresponding to $z = \mathrm{e}^{\mathrm{i}\phi} \in I$. Now $X^{(\epsilon)} = P_I \Delta^{(\epsilon)} P_I$, converging weakly to X, has kernel

$$X^{(\epsilon)}(\phi,\phi') = \chi_I(\phi)\left(\frac{1}{1+\mathrm{e}^{-(\mathrm{i}\phi-\mathrm{i}\phi'+\epsilon)/2}} - \frac{1}{2}\right)\chi_I(\phi')$$

and is no longer singular for $\epsilon \searrow 0$. Thus the kernel $X^{(0)}(\phi,\phi')$ that is obtained from $X^{(\epsilon)}$ by putting $\epsilon = 0$ is well-defined and hence

$$\lim_{\epsilon\searrow 0} \langle f, X^{(\epsilon)} g\rangle = \int_{-\pi}^{\pi} \frac{\mathrm{d}\phi}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}\phi'}{2\pi}\, \overline{f(\mathrm{e}^{\mathrm{i}\phi})} X^{(0)}(\phi,\phi') g(\mathrm{e}^{\mathrm{i}\phi'}), \qquad f,g \in L^2(S^1),$$

by the theorem of bounded convergence. It follows $X = X^{(0)} \in \mathcal{J}_2(L^2(S^1))$. Let $\tilde{\chi}_I$ be a smooth function on $[-\pi,\pi]$ which satisfies $\tilde{\chi}_I(\phi) = 1$ for $z = \mathrm{e}^{\mathrm{i}\phi} \in I$ and vanishes in a neighborhood of $\phi = \pm\pi$. We define

$$\tilde{X}(\phi,\phi') = \tilde{\chi}_I(\phi)\left(\frac{1}{1+\mathrm{e}^{-(\phi-\phi')/2}} - \frac{1}{2}\right)\tilde{\chi}_I(\phi')$$

such that $X = P_I \tilde{X} P_I$ and hence

$$\|X\|_1 = \|P_I \tilde{X} P_I\|_1 \le \|P_I\| \|\tilde{X}\|_1 \|P_I\| = \|\tilde{X}\|_1.$$

Since $\tilde{X}(\phi, \phi')$ is a smooth function in ϕ and ϕ' it has fast decreasing Fourier coefficients which coincide with matrix elements $\langle e_n, \tilde{X} e_m \rangle$, $n, m \in \mathbb{Z}$. This proves the statement $\|X\|_1 < \infty$. $\square$

In restriction to the local even algebra $\mathfrak{C}(I) = \mathcal{C}(\mathcal{K}(I), \Gamma)^+$, $I \in \mathcal{J}_\zeta$ we find by Lemma 3.8

$$(\pi_0 \oplus \pi_{\mathrm{v}})|_{\mathfrak{C}(I)} \approx \begin{cases} (\pi_{\mathrm{s}} \oplus \pi_{\mathrm{c}})|_{\mathfrak{C}(I)} & N = 2\ell \\ 2\pi_\sigma|_{\mathfrak{C}(I)} & N = 2\ell + 1 \end{cases}$$

Recall that $\pi_{\mathrm{v}} \simeq \pi_0 \circ \varrho_U$ with $U = 2|v\rangle\langle v| - \mathbf{1}$ as in Corollary 2.8. Choose $v \in \mathcal{K}(I')$. Then $\varrho_U(x) = x$ for $x \in \mathfrak{C}(I)$, hence π_0 and π_{v} are equivalent on $\mathfrak{C}(I)$. In the same way we obtain local equivalence of π_{s} and π_{c}. We conclude that indeed local normality holds for all sectors.

Theorem 3.9 *Restricted to local C^*-algebras $\mathfrak{C}(I)$, $I \in \mathcal{J}_\zeta$, the representations π_ξ are quasi-equivalent to the vacuum representation $\pi_0 = id$,*

$$\pi_\xi|_{\mathfrak{C}(I)} \approx \pi_0|_{\mathfrak{C}(I)}, \qquad I \in \mathcal{J}_\zeta, \quad \xi = \mathrm{v}, \mathrm{s}, \mathrm{c}, \sigma. \tag{3.17}$$

We have seen that we have an extension of our representations π_ξ to local von Neumann algebras $\mathcal{R}(I)$, $I \in \mathcal{J}_\zeta$, and thus to the quasilocal algebra $\mathcal{A}$ they generate. By unitary equivalence $\varrho_\xi \simeq \pi_\xi$ on $\mathfrak{C}(I_\zeta)$ we have an extension of ϱ_ξ to $\mathcal{A}$, too, $\xi = \mathrm{v}, \mathrm{s}, \mathrm{c}, \sigma$. Being localized in some $I \in \mathcal{J}_\zeta$, they inherit properties

$$\varrho_\xi(A) = A, \qquad A \in \mathcal{A}_\zeta(I'),$$

and

$$\varrho_\xi(\mathcal{R}(I_0)) \subset \mathcal{R}(I_0), \qquad I_0 \in \mathcal{J}_\zeta, \qquad I \subset I_0,$$

from the underlying C^*-algebras. So our endomorphisms ϱ_ξ are well-defined localized endomorphisms of $\mathcal{A}$ in the common sense. Moreover, they are transportable. This follows because the precosheaf $\{\mathcal{R}(I)\}$ is Möbius covariant. Hence $\mathcal{A}$ is covariant with respect to the subgroup of Möbius transformations leaving ζ invariant.

3.2.3 Fusion Rules

In this subsection we prove the fusion rules of our sectors $1, \mathrm{v}, \mathrm{s}, \mathrm{c}, \sigma$ in terms of unitary equivalence classes of localized endomorphisms $[\varrho] \equiv [\pi_0 \circ \varrho]$ (or,

equivalently, in terms of equivalence classes $[\pi]$ of representations π satisfying an DHR criterion). Because we deal with an Haag dual net of local von Neumann algebras, by standard arguments, it suffices to check a fusion rule $[\varrho_\xi \varrho_{\xi'}]$ for special representatives $\varrho_\xi \in [\varrho_\xi]$, $\varrho_{\xi'} \in [\varrho_{\xi'}]$. This will be done by our examples of Definitions 3.1 and 3.3. For instance, we clearly have $\mathrm{v} \times \mathrm{v} = 1$ for all $N \in \mathbb{N}$. Let us first consider the even case, $N = 2\ell$. By Corollary 2.8 we easily find $\mathrm{v} \times \mathrm{s} = \mathrm{c}$, $\mathrm{v} \times \mathrm{c} = \mathrm{s}$. Since V then is unitary and by Lemma 3.2 we have $\pi_{\mathrm{NS}} \circ \varrho_V^2 \simeq \pi_{\mathrm{NS}}$. Now π_{NS}, when restricted to $\mathfrak{C}(I_\zeta) \equiv \mathcal{C}(\mathcal{K}, \Gamma)^+$, decomposes into the basic and the vector representation. Hence only the possibilities $\mathrm{s} \times \mathrm{s} = 1$ or $\mathrm{s} \times \mathrm{s} = \mathrm{v}$ are left, i.e. we have to check whether $\pi_{\mathrm{NS}}^+ \circ \varrho_V^2$ is equivalent to π_{NS}^+ or π_{NS}^-, i.e. whether ϱ_{s} is a self-conjugate endomorphism or not. For $N = 2\ell$ the action of V in the $(2j-1)^{\mathrm{th}}$ and the $2j^{\mathrm{th}}$ component, $j = 1, 2, \ldots, \ell$, is the same as in the 1^{st} and the 2^{nd} component, respectively. So we can write the square $W = V^2$ as a product,

$$W = W_{1,2} W_{3,4} \cdots W_{N-1,N}$$

where $W_{1,2}$ acts as W in the first two components and as the identity in the others, etc. Since σ of Prop. 2.6 is multiplicative and clearly all $W_{2j-1,2j}$ lead to implementable automorphisms we have

$$\sigma(W) = \sigma(W_{1,2})\sigma(W_{3,4}) \cdots \sigma(W_{N-1,N}).$$

All $W_{2j-1,2j}$ are built in the same way, hence all the $\sigma(W_{2j-1,2j})$ are equal i.e. $\sigma(W) = \sigma(W_{1,2})^\ell$. Since σ takes only values ± 1 this is $\mathrm{s} \times \mathrm{s} = 1$ if ℓ is even. But for odd ℓ this reads $\sigma(W) = \sigma(W_{1,2})$. Thus we first check the case $N = 2$. If $\sigma(W_{1,2}) = +1$ then ϱ_{s} is self-conjugate, otherwise it is not self-conjugate, i.e. $\mathrm{s} \times \mathrm{s} = \mathrm{v}$. It is a result of Guido and Longo [34] that a conjugate morphism $\overline{\varrho}$ is given by

$$\overline{\varrho} = \mathrm{j} \circ \varrho \circ \mathrm{j}$$

where j is the antiautomorphism corresponding to the reflection $z \mapsto \overline{z}$ on the circle (PCT transformation). In our model, j is the extension of the antilinear Bogoliubov automorphism j_Θ,

$$\mathrm{j}_\Theta(B(f)) = B(\Theta f), \qquad \Theta f \equiv \Theta\left((f^i)_{i=1,2}\right) = \left(f^i_{\mathrm{refl}}\right)_{i=1,2},$$

where $f \in L^2(S^1; \mathbb{C}^2)$ and $f^i_{\mathrm{refl}}(z) = \overline{f^i(\overline{z})}$ for $z \in S^1$. So we have a candidate $\overline{\varrho_{\mathrm{s}}} \equiv \overline{\varrho_V} = \varrho_{\Theta V \Theta}$. It is quite obvious that $\Theta P_{I_\pm} \Theta = P_{I_\mp}$ and that $\Theta f^i_p = f^i_p$,

$p \in \frac{1}{2}\mathbb{Z}$, so it follows by antilinearity of Θ ($N = 2$)

$$\Theta V \Theta = -P_{I_-} + P_{I_+} + (-\mathrm{i}r^1 - \mathrm{i}R^1) - (t^2 - \mathrm{i}T^2).$$

It is not hard to see that this is

$$\Theta V \Theta = U_{1,2} V, \qquad U_{1,2} = 2|v^1_{1/2}\rangle\langle v^1_{1/2}| - \mathbf{1}, \qquad v^1_{1/2} = \frac{1}{\sqrt{2}}\,(f^1_{1/2} + f^1_{-1/2}).$$

Now $U_{1,2}$ is as in Corollary 2.8 so that we find $\mathrm{s} \times \mathrm{v} \times \mathrm{s} = \mathrm{s} \times \mathrm{c} = 1$ for $N = 2$. Hence $\sigma(W_{1,2}) = -1$, so it follows $\mathrm{s} \times \mathrm{c} = 1$ for all $N = 2\ell$ with ℓ odd. For the case $N = 2\ell + 1$ the situation is different because ϱ_V then is not an automorphism. As discussed at the end of Section 5, the representation $\pi_{\mathrm{NS}} \circ \varrho_V$ (and, of course, also $\pi_{\mathrm{NS}} \circ \varrho_U \varrho_V$) decomposes, in restriction to $\mathcal{C}(\mathcal{K}, \Gamma)^+$, into two equivalent irreducibles corresponding to the spinor sector σ. So we find at first $\mathrm{v} \times \sigma = \sigma$. Let us consider $\pi_{\mathrm{NS}} \circ \varrho_V^2$. We have $M_{V^2} = 2M_V = 2$, hence by Theorem 2.5 and Lemma 3.2 we conclude $\pi_{\mathrm{NS}} \circ \varrho_V^2 \simeq 2\pi_{\mathrm{NS}}$. In restriction to $\mathcal{C}(\mathcal{K}, \Gamma)^+$ this reads $\pi^+_{\mathrm{NS}} \circ \varrho_V^2 \oplus \pi^-_{\mathrm{NS}} \circ \varrho_V^2 \simeq 2(\pi^+_{\mathrm{NS}} \oplus \pi^-_{\mathrm{NS}})$. Our previous results admit only $\pi^+_{\mathrm{NS}} \circ \varrho_V^2 \simeq \pi^-_{\mathrm{NS}} \circ \varrho_V^2$ and hence we find $\sigma \times \sigma = 1 + \mathrm{v}$. Summarizing we rediscover the WZW fusion rules.

Theorem 3.10 *The DHR sector product reproduces the fusion rules (2.21) and (2.22).*

3.3 Remarks

We conclude this chapter with some general remarks on the presented analysis.

3.3.1 The Chiral Ising Model

Although the analysis of the $\mathfrak{so}(N)$ WZW models requires that $N \geq 7$ our analysis with fermions also works if one formally sets $N = 1$. In this case no current algebra appears; the chiral algebra (i.e. the unbounded observable algebra) consists just of a Virasoro algebra with central charge $c = 1/2$,

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{24}\, m(m^2 - 1)\, \delta_{m,-n};$$

then only one species of fermions is present so that

$$L_m = -\frac{1}{2} \sum_{r \in \mathbb{Z}+1/2} (r - \frac{m}{2})\, {:}b_r b_{m-r}{:}$$

in the Neveu-Schwarz sector, respectively

$$L_m = -\frac{1}{2} \sum_{n \in \mathbb{Z}} (n - \frac{m}{2}) :b_n b_{m-n}: + \delta_{m,0} \frac{1}{16} \mathbf{1}$$

in the Ramond sector. Under the action of the Virasoro algebra the Neveu-Schwarz sector splits into two different modules with conformal weights 0 and 1/2 (the "basic" and the "vector module") and the Ramond sector splits into two equivalent modules with conformal weight 1/16 (the "spinor module" σ). Thus the $N = 1$ case is just what is called the chiral Ising model which was first investigated in the algebraic framework in [41]. Indeed the analysis using localized endomorphisms of even CAR algebras reproduces the fusion rules (2.22) which are called the Ising fusion rules. This analysis has been carried out in [5] and is now included in our more general setting.

3.3.2 Discussion and Outlook

The main result of this chapter is the proof of the WZW fusion rules in terms of the DHR sector product. We believe that this result is remarkable for two reasons. Firstly, non-trivial CFT models could be incorporated mathematically rigorously in the DHR framework, and expected correspondences between CFT and AQFT could be established. Secondly, the proof is completely independent of the methods that are conventionally used in CFT to derive fusion rules. This is noteworthy since methods like operator product expansions are not all under sufficient mathematical control. However, our results are based on the fact that the $\mathbb{Z}_2$-invariant fermion algebras are identified to be the bounded observable algebras of the level 1 $\mathfrak{so}(N)$ WZW models. Therefore difficulties arise when one tries to generalize this program to other models. As we will see in the next chapter, already at level 2 there is no longer a gauge invariant subalgebra of the fermion algebra that can be identified as bounded WZW observable algebra. Also to $\mathfrak{su}(N)$ WZW models (at arbitrary level) we cannot directly translate this program.

In order to incorporate other WZW models into the DHR framework, Wassermann's loop group approach may be the more suitable one. However, it would be desirable to find the relevant localized endomorphisms of the operator algebras in the vacuum representation so that the fusion rules could be directly derived in terms of the DHR sector product instead of using Connes fusion.

Chapter 4

$\mathfrak{so}(N)$ Wess-Zumino-Witten Models at Level 2

In this chapter we tackle the problem to identify the $\widehat{\mathfrak{so}}(N)_2$ highest weight modules appearing in the doubled Neveu-Schwarz sector $\hat{\mathcal{H}}_{\mathrm{NS}} = \mathcal{H}_{\mathrm{NS}} \otimes \mathcal{H}_{\mathrm{NS}}$, the "big Fock space", where $\mathcal{H}_{\mathrm{NS}}$ is the Neveu-Schwarz sector of the level 1 theory. We discuss the realization of $\widehat{\mathfrak{so}}(N) \rtimes \mathfrak{Vir}$ in $\hat{\mathcal{H}}_{\mathrm{NS}}$. Crucial for our construction is the application of the DHR theory to a fermionic field algebra acting in the big Fock space; we introduce the DHR gauge group $O(2)$ which leaves the chiral algebra invariant. The decomposition of the big Fock space into the sectors of the gauge invariant fermion algebra turns out to be helpful for the construction of the simultaneous highest weight vectors of $\widehat{\mathfrak{so}}(N)_2$ and the coset Virasoro algebra $\mathfrak{Vir}^{\mathrm{c}}$. A detailed analysis of the characters ends up with the complete decomposition of the big Fock space into tensor products of irreducible $\widehat{\mathfrak{so}}(N)_2$ and $\mathfrak{Vir}^{\mathrm{c}}$ highest weight modules. This analysis is based on [8].

4.1 The Doubled Neveu-Schwarz Sector

In this section we introduce the big Fock space $\hat{\mathcal{H}}_{\mathrm{NS}}$ and the "doubled" fermion algebra acting on it. In view of the fact that the generators of $\widehat{\mathfrak{so}}(N) \rtimes \mathfrak{Vir}$ as well as those of the Virasoro algebra $\mathfrak{Vir}^{\mathrm{c}}$ of the coset theory $(\widehat{\mathfrak{so}}(N)_1 \oplus \widehat{\mathfrak{so}}(N)_1)/\widehat{\mathfrak{so}}(N)_2$ are both invariant under the gauge group $O(2)$ we decompose the big Fock space into the sectors of the gauge invariant fermion algebra. According to the results of the DHR theory, the representation

theory of the gauge group determines this decomposition.

4.1.1 The Doubled CAR Algebra

We are interested in the theory that is obtained by doubling the Neveu-Schwarz fermions of the type described in Chapter 3. Thus in addition to the $\mathfrak{so}(N)$ index i the fermion modes will now be labelled by a “flavor” index $q = 1, 2$. To describe this theory, we define

$$\hat{\mathcal{K}} = \mathcal{K} \oplus \mathcal{K}\,, \quad \hat{\Gamma} = \Gamma \oplus \Gamma \quad \text{and} \quad \hat{P}_{\text{NS}} = P_{\text{NS}} \oplus P_{\text{NS}}\,,$$

or, alternatively,

$$\hat{\mathcal{K}} = \mathcal{K} \otimes \mathbb{C}^2, \quad \hat{\Gamma} = \Gamma \otimes \Gamma_2 \quad \text{and} \quad \hat{P}_{\text{NS}} = P_{\text{NS}} \otimes \mathbb{1}_2\,,$$

where Γ_2 denotes the canonical complex conjugation in $\mathbb{C}^2$. Further, for any $f \in \mathcal{K}$ we define the elements

$$B^q(f) = B(f \otimes v^q)\,, \qquad q = 1, 2\,,$$

of $\mathcal{C}(\hat{\mathcal{K}}, \hat{\Gamma})$, where v^q denote the canonical unit vectors of $\mathbb{C}^2$. We denote by $(\hat{\mathcal{H}}_{\text{NS}}, \hat{\pi}_{\text{NS}}, |\hat{\Omega}_{\text{NS}}\rangle)$ the GNS representation associated to the Fock state $\omega_{\hat{P}_{\text{NS}}}$ of $\mathcal{C}(\hat{\mathcal{K}}, \hat{\Gamma})$. We then define the Fourier modes

$$b_r^{i;q} = \hat{\pi}_{\text{NS}}(B^q(e_r^i)) \tag{4.1}$$

for $i = 1, 2, \dots, N$, $q = 1, 2$ and $r \in \mathbb{Z} + \frac{1}{2}$. The Fourier modes $b_r^{i;q}$ generate a CAR algebra with relations

$$\{b_r^{i;p}, b_s^{j;q}\} \;=\; \delta_{p,q}\, \delta_{i,j}\, \delta_{r+s,0}\, \mathbf{1}$$

and $(b_r^{i;q})^* = b_{-r}^{i;q}$. The modes $b_r^{i;q}$ with positive index r act as annihilation operators in $\hat{\mathcal{H}}_{\text{NS}}$, i.e. for all $q = 1, 2$ and all $i = 1, 2, \dots, N$ we have

$$b_r^{i;q}\, |\hat{\Omega}_{\text{NS}}\rangle = 0 \qquad \text{for } r \in \mathbb{N}_0 + \tfrac{1}{2}\,.$$

4.1.2 Realization of $\widehat{\mathfrak{so}}(N) \rtimes \mathfrak{Vir}$ at Level 2

Given the fermion modes (4.1), one defines again normal ordering

$$:\!b_r^{i;p} b_s^{j;q}\!: \,= \begin{cases} \;\; b_r^{i;p} b_s^{j;q} & r < 0 \\ -b_s^{j;q} b_r^{i;p} & r > 0 \end{cases}, \qquad r, s \in \mathbb{Z} + \tfrac{1}{2}\,,$$

sums over their normal-ordered bilinears

$$B_m^{i,j;q} = \frac{1}{2} \sum_{r \in \mathbb{Z}+1/2} :b_r^{i;q} b_{m-r}^{j;q}: \,, \qquad q = 1, 2$$

and current operators

$$J_m^{i,j} = \mathrm{i} \sum_{q=1}^{2} \big[B_m^{i,j;q} - B_m^{j,i;q} \big] \,, \tag{4.2}$$

for $i, j = 1, 2, \dots, N$. Quite analogous to the situation in $\mathcal{H}_{\mathrm{NS}}$, all the unbounded expressions we introduce here possess an invariant dense domain $\hat{\mathcal{H}}_{\mathrm{NS}}^{\mathrm{fin}} \subset \hat{\mathcal{H}}_{\mathrm{NS}}$ spanned by finite energy vectors. One checks by direct computation that

$$[J_m^{i,j}, b_r^{k;q}] = \mathrm{i}\, (\delta_{j,k} b_{r+m}^{i;q} - \delta_{i,k} b_{r+m}^{j;q}) \,,$$

and

$$\begin{aligned} [J_m^{i,j}, J_n^{k,l}] = \mathrm{i}\, (\delta_{j,k} J_{m+n}^{i,l} + \delta_{i,l} J_{m+n}^{j,k} - \delta_{j,l} J_{m+n}^{i,k} - \delta_{i,k} J_{m+n}^{j,l}) + \\ + 2\, m\, \delta_{m,-n} (\delta_{i,k}\delta_{j,l} - \delta_{i,l}\delta_{j,k}) \,. \end{aligned} \tag{4.3}$$

According to (4.3) (compare also (2.19)), the $J_m^{i,j}$ with $i < j$ provide a basis for the affine Lie algebra $\widehat{\mathfrak{so}}(N)$ at fixed value $k^\vee = 2$ of the level. That the level of $\widehat{\mathfrak{so}}(N)$ has the value 2 is of course a consequence of the summation over two species of fermions in (4.2); while for a single fermion we obtained the Lie algebra $\widehat{\mathfrak{so}}(N)$ at level 1 we now observe that the $J_m^{i,j}$ correspond via second quantization to operators $\tau_m^{i,j} \otimes \mathbb{1}_2$ on $\hat{\mathcal{K}}$, so that the Schwinger term (2.10) is doubled now.

Recall the Chevalley basis of the affine Lie algebra $\widehat{\mathfrak{so}}(N)_2$; the Cartan subalgebra generators are $\mathcal{H}^j = J_0^{2j-1,2j}$ for $j = 1, 2, \dots, \ell$, and the Chevalley generators $\mathcal{E}_\pm^j$ are given by

$$\mathcal{E}_\pm^j = \pm J_0(t_{\pm,\mp}^{j,j+1}) \qquad \text{for } j = 1, 2, \dots, \ell-1 \,,$$

$$\mathcal{E}_\pm^0 = \pm J_{\pm 1}(t_{\mp,\mp}^{1,2}) \,, \qquad \mathcal{E}_\pm^\ell = \begin{cases} \pm J_0(t_{\pm,\pm}^{\ell-1,\ell}) & \text{for } N = 2\ell \,, \\ \pm J_0(t_\pm^\ell) & \text{for } N = 2\ell+1 \,, \end{cases}$$

where

$$J_m(t_{\varepsilon,\eta}^{i,j}) = \frac{1}{2} (\varepsilon J_m^{2i,2j-1} + \eta J_m^{2i-1,2j}) + \frac{\mathrm{i}}{2} (J_m^{2i-1,2j-1} - \varepsilon\eta J_m^{2i,2j}) \,,$$

$$J_m(t_\varepsilon^j) = -\frac{1}{\sqrt{2}} (\varepsilon J_m^{2j-1,2\ell+1} - \mathrm{i} J_m^{2j,2\ell+1})$$

for $i, j = 1, 2, \dots, \ell$ and $\varepsilon, \eta = \pm 1$.

The generators of the associated Virasoro algebra, i.e. the Laurent modes of the stress energy tensor of the WZW theory, will be denoted by L_m. In our particular case, Sugawara's formula reads

$$L_m = \frac{1}{2N} \sum_{1 \le i < j \le N} \sum_{n \in \mathbb{Z}} :J^{i,j}_n J^{i,j}_{m-n}:$$

where the normal ordering of the current operators is defined by

$$:J^{i,j}_m J^{i,j}_n: = \begin{cases} J^{i,j}_m J^{i,j}_n & m < 0 \\ J^{i,j}_n J^{i,j}_m & m \ge 0 \, . \end{cases}$$

This Virasoro algebra has central charge $c = N - 1$. Also, we denote by L^{NS}_m the Laurent components of the canonical stress energy tensor of the fermion theory in the big Fock space (i.e. the Sugawara operators associated to the semisimple Lie algebra $\mathfrak{so}(N) \oplus \mathfrak{so}(N)$, compare Chapter 1, Subsection 1.2.3),

$$L^{\text{NS}}_m = L^{(1)}_m + L^{(2)}_m \qquad \text{with} \quad L^{(q)}_m = -\tfrac{1}{2} \sum_{i=1}^{N} \sum_{r \in \mathbb{Z}+1/2} (r - \tfrac{m}{2}) :b^{i;q}_r b^{i;q}_{m-r}: \, . \quad (4.4)$$

Thus in particular

$$L^{(q)}_0 = \sum_{i=1}^{N} \sum_{r \in \mathbb{N}_0 + 1/2} r \, b^{i;q}_{-r} b^{i;q}_r \, .$$

Note that the L^{NS}_m correspond via second quantization to operators $\lambda_m \otimes \mathbb{1}_2$ on $\hat{\mathcal{K}}$ but not the L_m. Although the L^{NS}_m satisfy the same commutation relations with the current operators as the Sugawara operators L_m do,

$$[L^{\text{NS}}_m, J^{i,j}_n] = [L_m, J^{i,j}_n] = -n \, J^{i,j}_{m+n} \, ,$$

they generate a Virasoro algebra with central charge $c^{\text{NS}} = N$. This implies that the coset Virasoro operators

$$L^{\text{c}}_m = L^{\text{NS}}_m - L_m$$

commute with the current operators $J^{i,j}_n$ and generate the coset Virasoro algebra $\mathfrak{Vir}^{\text{c}}$ with central charge $c^{\text{c}} = c^{\text{NS}} - c = 1$.

4.1.3 DHR Theory with Gauge Group $O(2)$

The group $O(2)$ is generated by $GL\,(2;\mathbb{C})$ matrices

$$\gamma_t = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}, \quad t\in\mathbb{R}\,, \qquad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Correspondingly, we define Bogoliubov operators in $\mathcal{Q}(\hat{\mathcal{K}},\hat{\Gamma})$,

$$U(\gamma_t) = \mathbf{1}\otimes\gamma_t\,, \qquad U(\eta) = \mathbf{1}\otimes\eta$$

acting on $\hat{\mathcal{K}} = \mathcal{K}\otimes\mathbb{C}^2$; they induce Bogoliubov automorphisms $\varrho_{U(\gamma_t)}$, $\varrho_{U(\eta)}$, respectively. These automorphisms fulfill

$$\varrho_{U(\gamma_t)}(B^1(f)) = \cos(t)\,B^1(f) - \sin(t)\,B^2(f)\,,$$
$$\varrho_{U(\gamma_t)}(B^2(f)) = \sin(t)\,B^1(f) + \cos(t)\,B^2(f)$$

and

$$\varrho_{U(\eta)}(B^1(f)) = B^1(f)\,, \qquad \varrho_{U(\eta)}(B^2(f)) = -B^2(f)\,.$$

The invariance of the Fock state $\omega_{\hat{P}_{\rm NS}}$ reads now

$$\omega_{\hat{P}_{\rm NS}}\circ\varrho_{U(\gamma_t)} = \omega_{\hat{P}_{\rm NS}} = \omega_{\hat{P}_{\rm NS}}\circ\varrho_{U(\eta)}\,,$$

and hence there is a unitary (strongly continuous) representation Q of $O(2)$ by certain implementers $Q(\gamma_t)\equiv Q_{\hat{P}_{\rm NS}}(U(\gamma_t))$ and $Q(\eta)\equiv Q_{\hat{P}_{\rm NS}}(U(\eta))$ in $\mathfrak{B}(\hat{\mathcal{H}}_{\rm NS})$ which satisfy

$$Q(\gamma_t)\,|\hat{\Omega}_{\rm NS}\rangle = |\hat{\Omega}_{\rm NS}\rangle = Q(\eta)\,|\hat{\Omega}_{\rm NS}\rangle\,,$$

and the action of $\varrho_{U(\gamma_t)}$ and $\varrho_{U(\eta)}$ extends to $\mathfrak{B}(\hat{\mathcal{H}}_{\rm NS})$.

The inequivalent finite-dimensional irreducible representations of $O(2)$ are the following. Besides the identity Φ_0 with $\Phi_0(\cdot) = 1$ and another one-dimensional representation Φ_J with

$$\Phi_J(\gamma_t) = 1\,, \qquad \Phi_J(\eta) = -1\,, \tag{4.5}$$

there are only two-dimensional representations $\Phi_{[m]}$ with $m = 1,2,\ldots$; their representation matrices are

$$\Phi_{[m]}(\gamma_t) = \begin{pmatrix} \mathrm{e}^{imt} & 0 \\ 0 & \mathrm{e}^{-imt} \end{pmatrix}, \qquad \Phi_{[m]}(\eta) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.6}$$

The tensor product decompositions of these representations read

$$\begin{aligned} &\Phi_J \times \Phi_J = \Phi_0 \,, \qquad\qquad \Phi_J \times \Phi_{[m]} = \Phi_{[m]} \,, \\ &\Phi_{[m]} \times \Phi_{[n]} = \Phi_{[|m-n|]} + \Phi_{[m+n]} \qquad \text{for } m \neq n \,, \\ &\Phi_{[n]} \times \Phi_{[n]} = \Phi_0 + \Phi_J + \Phi_{[2n]} \,. \end{aligned} \tag{4.7}$$

Field and observable algebras of the fermion theory are described as follows. Choose a point $\zeta \in S^1$ on the circle and denote by $\mathcal{J}_\zeta$ the set of those open intervals $I \subset S^1$ whose closures do not contain ζ. For $I \in \mathcal{J}_\zeta$ let $\mathcal{K}(I)$ be the subspace of functions having support in I. Correspondingly, define $\hat{\mathcal{K}}(I) = \mathcal{K}(I) \otimes \mathbb{C}^2$. The local field algebras $\mathfrak{F}(I)$ are then defined to be the von Neumann algebras

$$\mathfrak{F}(I) = \hat{\pi}_{\mathrm{NS}}(\mathcal{C}(\hat{\mathcal{K}}(I), \hat{\Gamma}))'' \,,$$

and the global field algebra $\mathfrak{F}$ is the C^*-algebra that is defined as the norm closure of the union of the local algebras,

$$\mathfrak{F} = \overline{\bigcup_{I \in \mathcal{J}_\zeta} \mathfrak{F}(I)} \,.$$

The group $O(2)$ acts on the field algebra as a gauge group in the sense of Doplicher, Haag and Roberts [15]. This a subgroup of the automorphism group of $\mathfrak{F}(I)$ respectively $\mathfrak{F}$ such that the observables are precisely the gauge invariant fields (compare Chapter 1, Subsection 1.1.1). Therefore the local observable algebras $\mathfrak{A}(I)$ and the global (or quasi-local) observable algebra $\mathfrak{A}$ are defined as $O(2)$-invariant part of the field algebras,

$$\mathfrak{A}(I) = \mathfrak{F}(I) \cap Q(O(2))'$$

and

$$\mathfrak{A} = \overline{\bigcup_{I \in \mathcal{J}_\zeta} \mathfrak{A}(I)} \,.$$

At level 2 the algebra $\mathfrak{A}$ does not coincide with the observable algebra $\mathfrak{A}_{\mathrm{WZW}}$ of bounded operators which is associated to the WZW theory. Indeed we will see that each irreducible $\mathfrak{A}$-sector is highly reducible under the action of the observable algebra $\mathfrak{A}_{\mathrm{WZW}}$. Nevertheless, owing to $\mathfrak{A}_{\mathrm{WZW}} \subset \mathfrak{A}$ the representation theory of $\mathfrak{A}$ turns out to be crucial for our analysis of the decomposition of the big Fock space into tensor products of highest weight modules of the level 2 chiral algebra and of the coset Virasoro algebra.

For the construction of the highest weight vectors within the $\mathfrak{A}$-sectors it is convenient to work with the unbounded operators of $\widehat{\mathfrak{so}}(N)$ (instead of the bounded elements of $\mathfrak{A}_{\rm WZW}$) and of the Virasoro algebra that is associated to $\widehat{\mathfrak{so}}(N)_2$ by the Sugawara formula.

The Bogoliubov automorphisms act as rotations on the flavor index q of the fermions. As a consequence, they leave expressions of the form

$$\sum_{q=1}^{2} B^q(f)\, B^q(g) \qquad (f,g\in\mathcal{K})$$

invariant. In particular, it can be easily read off their definition that the current operators $J^{i,j}_m$ remain invariant under the action of the gauge group $O(2)$. Similarly, owing to the summation on q in the bilinear expression (4.4), the Virasoro generators $L^{\rm NS}_m$ are $O(2)$-invariant, too. This implies that the coset Virasoro operators $L^{\rm c}_m$ are gauge invariant as well. Therefore neither the current operators of $\widehat{\mathfrak{so}}(N)_2$ nor the elements of $\mathfrak{Vir}^{\rm c}$ make transitions between the sectors of $\mathfrak{A}$ (and hence in particular $\mathfrak{A}_{\rm WZW}\subset\mathfrak{A}$). For the decomposition of the big Fock space $\hat{\mathcal{H}}_{\rm NS}$ into their (highest weight) modules it may thus be helpful to decompose $\hat{\mathcal{H}}_{\rm NS}$ first into the sectors of $\mathfrak{A}$. Employing the results of [15, 19], we arrive at

$$\hat{\mathcal{H}}_{\rm NS}=\mathcal{H}_0\oplus\mathcal{H}_J\oplus\bigoplus_{m=1}^{\infty}(\mathcal{H}_{[m]}\otimes H_{[m]})\,.$$

Here $\mathcal{H}_0$, $\mathcal{H}_J$ and $\mathcal{H}_{[m]}$ carry mutually inequivalent irreducible representations of $\mathfrak{A}$; vectors in $\mathcal{H}_0,\mathcal{H}_J$ transform according to the two inequivalent one-dimensional irreducible representations Φ_0 and Φ_J of the gauge group $O(2)$, respectively, and the $H_{[m]}\simeq\mathbb{C}^2$ carry the inequivalent two-dimensional irreducible $O(2)$-representations $\Phi_{[m]}$. Later we will also use the notation

$$\mathcal{H}_{[m]}\otimes H_{[m]}=\mathcal{H}^{+}_{[m]}\oplus\mathcal{H}^{-}_{[m]}\,,$$

where by definition, $Q(\gamma_t)$ acts on $\mathcal{H}^{\pm}_{[m]}$ as multiplication with ${\rm e}^{\pm {\rm i}mt}$.

At level 1 we have $\mathcal{H}_{\rm NS}=\mathcal{H}^{(1)}_{\circ}\oplus\mathcal{H}^{(1)}_{\rm v}$, and hence at level 2 we can write

$$\hat{\mathcal{H}}_{\rm NS}=(\mathcal{H}^{(1)}_{\circ}\otimes\mathcal{H}^{(1)}_{\circ})\oplus(\mathcal{H}^{(1)}_{\circ}\otimes\mathcal{H}^{(1)}_{\rm v})\oplus(\mathcal{H}^{(1)}_{\rm v}\otimes\mathcal{H}^{(1)}_{\circ})\oplus(\mathcal{H}^{(1)}_{\rm v}\otimes\mathcal{H}^{(1)}_{\rm v})\,. \quad (4.8)$$

The four summands in this decomposition can be characterized as the common eigenspaces with respect to the "fermion flips" $Q(\gamma_\pi\eta)$ and $Q(\eta)$, namely

those associated to the pairs $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$ of eigenvalues, respectively. By comparison with the action (4.5) and (4.6) of $O(2)$ on the $\mathfrak{A}$ sectors, it follows that we can decompose the tensor products appearing in (4.8) as

$$\begin{gathered}\mathcal{H}^{(1)}_{\circ} \otimes \mathcal{H}^{(1)}_{\circ} = \mathcal{H}_0 \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_{[2n]}\,, \qquad \mathcal{H}^{(1)}_{\mathrm{v}} \otimes \mathcal{H}^{(1)}_{\mathrm{v}} = \mathcal{H}_J \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_{[2n]}\,, \\ \mathcal{H}^{(1)}_{\circ} \otimes \mathcal{H}^{(1)}_{\mathrm{v}} = \bigoplus_{n=0}^{\infty} \mathcal{H}_{[2n+1]} = \mathcal{H}^{(1)}_{\mathrm{v}} \otimes \mathcal{H}^{(1)}_{\circ}\,.\end{gathered} \tag{4.9}$$

Later we will employ the representation theory of the gauge group $O(2)$, and in particular the decomposition (4.9), to obtain also simple formulae for the characters of the level 2 modules in the big Fock space. As further input, we will need some information about the relevant coset conformal field theories.

4.2 Highest Weight Vectors

Recall that a highest weight vector $|\Phi_\Lambda\rangle$ of $\widehat{\mathfrak{so}}(N)_2$ with highest weight Λ is characterized by the following properties. Firstly, it is annihilated by the step operators associated to the horizontal positive roots, i.e. for $1 \le i < j \le \ell$ and $\varepsilon = \pm 1$ one has

$$J_0(t^{i,j}_{+,\varepsilon})\,|\Phi_\Lambda\rangle = 0\,, \qquad \text{and also} \quad J_0(t^k_+)\,|\Phi_\Lambda\rangle = 0 \ \text{ for } N = 2\ell+1\,;$$

secondly, it is also annihilated by the step operators with positive grade, i.e. for $m > 0$, $i, j = 1, 2, \dots, \ell$ and $\varepsilon, \eta = \pm 1$ it satisfies

$$J_m(t^{i,j}_{\varepsilon,\eta})\,|\Phi_\Lambda\rangle = 0\,, \qquad \text{and also} \quad J_m(t^k_\varepsilon)\,|\Phi_\Lambda\rangle = 0 \ \text{ for } N = 2\ell+1\,;$$

(note that the above conditions are equivalent to the requirement $\mathcal{E}^j_+\,|\Phi_\Lambda\rangle = 0$, $j = 0, 1, \dots, \ell$) and thirdly, $|\Phi_\Lambda\rangle$ is an eigenvector of the Cartan subalgebra,

$$\mathcal{H}^k\,|\Phi_\Lambda\rangle = \Lambda^k\,|\Phi_\Lambda\rangle$$

for $k = 1, 2, \dots, \ell$.

We will exploit the decomposition of $\hat{\mathcal{H}}_{\mathrm{NS}}$ into irreducible $\mathfrak{A}$ sectors to identify the highest weight vectors of $\widehat{\mathfrak{so}}(N)_2$. Indeed, in each sector $\mathcal{H}_0$, $\mathcal{H}_J$ and $\mathcal{H}^{\pm}_{[m]}$ we find distinguished states which are highest weight vectors for both $\widehat{\mathfrak{so}}(N)_2$ and the coset Virasoro algebra.

4.2.1 The Combinations $x_r^{j,\pm}$ and $\bar{x}_r^{j,\pm}$

For the construction of the simultaneous highest weight vectors of $\widehat{\mathfrak{so}}(N)_2$ and $\mathfrak{Vir}^c$ it is convenient to introduce new creation and annihilation operators in terms of linear combinations of the $b_r^{i;q}$. We define

$$x_r^{j,\pm} = \frac{1}{\sqrt{2}}\,(c_r^{j,+} \pm \mathrm{i}\bar{c}_r^{j,+})\,, \qquad \bar{x}_r^{j,\pm} = \frac{1}{\sqrt{2}}\,(c_r^{j,-} \pm \mathrm{i}\bar{c}_r^{j,-})\,,$$

for $j = 1, 2, \ldots, \ell$, where

$$c_r^{j,\pm} = \frac{1}{\sqrt{2}}\,(b_r^{2j;1} \pm \mathrm{i}b_r^{2j-1;1})\,, \qquad \bar{c}_r^{j,\pm} = \frac{1}{\sqrt{2}}\,(b_r^{2j;2} \pm \mathrm{i}b_r^{2j-1;2})\,,$$

and also, for $N = 2\ell + 1$,

$$\bar{x}_r^{\ell+1,\pm} = \frac{1}{\sqrt{2}}\,(b_r^{2\ell+1;1} \pm \mathrm{i}b_r^{2\ell+1;2})\,.$$

Further, we set

$$\begin{aligned} X_r^{j,\pm} &= x_r^{j,\pm}\,x_r^{j-1,\pm}\cdots x_r^{1,\pm} && \text{for } j = 1, 2, \ldots, \ell\,, \\ \bar{X}_r^{j,\pm} &= \bar{x}_r^{j+1,\pm}\,\bar{x}_r^{j+2,\pm}\cdots \bar{x}_r^{\ell,\pm} && \text{for } j = 0, 1, \ldots, \ell - 1\,, \end{aligned}$$

and $\bar{X}_r^{\ell,\pm} = \mathbf{1}$. By direct calculation, we obtain

$$[\mathcal{H}^j, x_r^{k,\pm}] = \delta_{j,k}\,x_r^{k,\pm}\,, \qquad [\mathcal{H}^j, \bar{x}_r^{k,\pm}] = -\delta_{j,k}\,\bar{x}_r^{k,\pm}\,, \tag{4.10}$$

for all $j, k = 1, 2, \ldots, \ell$, and similarly, for $N = 2\ell + 1$,

$$[\mathcal{H}^j, \bar{x}_r^{\ell+1,\pm}] = 0 \tag{4.11}$$

for all $j = 1, 2, \ldots, \ell$. To find also the commutators of the fermion modes with the raising operators $\mathcal{E}_+^j$, we first compute

$$[J_m(t_{\varepsilon,\eta}^{i,j}), c_r^{k,\pm}] = \tfrac{1}{2}\,\varepsilon\,(\eta \mp 1)\,\delta_{j,k}\,c_{m+r}^{i,\varepsilon} - \tfrac{1}{2}\,\eta\,(\varepsilon \mp 1)\,\delta_{i,k}\,c_{m+r}^{j,\eta}\,.$$

Analogous relations hold for $[J_m(t_{\varepsilon,\eta}^{i,j}), \bar{c}_r^{k,\pm}]$. When $N = 2\ell + 1$ we have in addition the relation $[J_m(t_{\varepsilon,\eta}^{i,j}), b_r^{2\ell+1;q}] = 0$ and

$$\begin{aligned} &[J_m(t_+^j), c_r^{k,\pm}] = \mp\,\delta_{j,k}\,b_{m+r}^{2\ell+1;1}\,, \qquad [J_m(t_-^j), c_r^{k,\pm}] = 0\,, \\ &[J_m(t_\pm^j), b_r^{2\ell+1;1}] = -\,c_{m+r}^{j,\pm}\,, \end{aligned}$$

and similar relations for $[J_m(t_+^j), \bar{c}_r^{k,\pm}]$, $[J_m(t_-^j), \bar{c}_r^{k,\pm}]$ and $[J_m(t_\pm^j), b_r^{2\ell+1;2}]$. From these results we learn that

$$
\begin{aligned}
&[\mathcal{E}_+^0, x_r^{k,\pm}] = \delta_{k,2}\, \bar{x}_{r+1}^{1,\pm} - \delta_{k,1}\, \bar{x}_{r+1}^{2,\pm}\,, \qquad [\mathcal{E}_+^0, \bar{x}_r^{k,\pm}] = 0\,,\\
&[\mathcal{E}_+^j, x_r^{k,\pm}] = -\delta_{k,j+1}\, x_r^{j,\pm}\,, \quad [\mathcal{E}_+^j, \bar{x}_r^{k,\pm}] = \delta_{k,j}\, \bar{x}_r^{j+1,\pm} \quad \text{for } j = 1, 2, \ldots, \ell - 1\,,\\
&[\mathcal{E}_+^\ell, x_r^{k,\pm}] = 0\,, \quad [\mathcal{E}_+^\ell, \bar{x}_r^{k,\pm}] = \begin{cases} \delta_{k,\ell}\, x_r^{\ell-1,\pm} - \delta_{k,\ell-1}\, x_r^{\ell,\pm} & \text{for } N = 2\ell\,,\\ \delta_{k,\ell}\, \bar{x}_r^{\ell+1,\pm} - \delta_{k,\ell+1}\, x_r^{\ell,\pm} & \text{for } N = 2\ell + 1\,. \end{cases}
\end{aligned}
$$

Taking into account that $(x_r^{j,\pm})^2 = (\bar{x}_r^{j,\pm})^2 = 0$ and $(x_r^{j,\pm})^* = \bar{x}_{-r}^{j,\mp}$, these relations imply that

$$
\begin{aligned}
&[\mathcal{E}_+^j, X_r^{k,\pm}] = 0 \qquad \text{for } j = 1, 2, \ldots, \ell\,,\\
&[\mathcal{E}_+^j, \bar{X}_r^{k,\pm}] = 0 \qquad \text{for } j = 1, 2, \ldots, \ell - 1\,.
\end{aligned}
\tag{4.12}
$$

For $j = \ell$ we have instead

$$
\begin{aligned}
&[\mathcal{E}_+^\ell, \bar{X}_r^{k,\pm}] \cdot X_r^{\ell,\pm} = 0 \qquad \text{for } N = 2\ell\,,\\
&[\mathcal{E}_+^\ell, \bar{X}_r^{k,\pm}] \cdot \bar{x}_r^{\ell+1,\pm} = 0\,, \qquad [\mathcal{E}_+^\ell, \bar{x}_r^{\ell+1,\pm}] \cdot X_r^{\ell,\pm} = 0 \quad \text{for } N = 2\ell + 1\,.
\end{aligned}
\tag{4.13}
$$

Finally, for $j = 0$ we find

$$
[\mathcal{E}_+^0, \bar{X}_r^{k,\pm}] = 0\,, \qquad [\mathcal{E}_+^0, X_r^{k,\pm}] \cdot \bar{X}_{r+1}^{0,\pm} = 0\,. \tag{4.14}
$$

4.2.2 Simultaneous Highest Weight Vectors of $\widehat{\mathfrak{so}}(N)_2$ and $\mathfrak{Vir}^{\mathrm{c}}$

Now we are in a position to define a lot of vectors which will be proven to be highest weight states for both, the affine Lie algebra $\widehat{\mathfrak{so}}(N)_2$ and the coset Virasoro algebra.

Definition 4.1 *For $n = 0, 1, 2, \ldots$ we set*

$$
|\Omega_{[j]}^{n,\pm}\rangle = X_{-n-1/2}^{j,\pm}\, |\Omega_\circ^{n,\pm}\rangle\,, \qquad \text{for } j = 1, 2, \ldots, \ell\,, \tag{4.15}
$$

as well as

$$
|\overline{\Omega}_{[j]}^{n,\pm}\rangle = \begin{cases} \bar{X}_{-n-1/2}^{j,\pm} X_{-n-1/2}^{\ell,\pm}\, |\Omega_\circ^{n,\pm}\rangle & \text{for } N = 2\ell\,,\ j = 1, 2, \ldots, \ell - 1\,,\\ \bar{X}_{-n-1/2}^{j,\pm} \bar{x}_{-n-1/2}^{\ell+1,\pm} X_{-n-1/2}^{\ell,\pm}\, |\Omega_\circ^{n,\pm}\rangle & \text{for } N = 2\ell + 1\,,\ j = 1, 2, \ldots, \ell\,, \end{cases} \tag{4.16}
$$

where

$$|\Omega_\circ^{n+1,\pm}\rangle = \begin{cases} \bar{X}^{0,\pm}_{-n-1/2} X^{\ell,\pm}_{-n-1/2} \,|\Omega_\circ^{n,\pm}\rangle & \text{for } N = 2\ell\,, \\ \bar{X}^{0,\pm}_{-n-1/2} \bar{x}^{\ell+1,\pm}_{-n-1/2} X^{\ell,\pm}_{-n-1/2} \,|\Omega_\circ^{n,\pm}\rangle & \text{for } N = 2\ell+1\,, \end{cases} \tag{4.17}$$

recursively, and

$$|\Omega_\circ^{0,\pm}\rangle \equiv |\Omega_\circ\rangle = |\hat{\Omega}_{\rm NS}\rangle\,. \tag{4.18}$$

Further, we set

$$\begin{aligned} \pm|\Omega_{\rm v}^{0,\pm}\rangle &\equiv |\Omega_{\rm v}\rangle = x^{1,+}_{-1/2} x^{1,-}_{-1/2}\,|\Omega_\circ\rangle\,, \\ |\Omega_{\rm v}^{n,\pm}\rangle &= x^{1,\pm}_{-n-1/2} x^{1,\mp}_{n-1/2}\,|\Omega_\circ^{n,\pm}\rangle\,, \quad n = 1,2,\ldots\,, \end{aligned} \tag{4.19}$$

and, for $N = 2\ell$,

$$|\Omega_{\rm s}^{n,\pm}\rangle = |\Omega_{[\ell]}^{n,\pm}\rangle\,, \qquad |\Omega_{\rm c}^{n,\pm}\rangle = \bar{x}^{\ell,\pm}_{-n-1/2} \bar{x}^{\ell,\mp}_{n+1/2}\,|\Omega_{\rm s}^{n,\pm}\rangle\,. \tag{4.20}$$

Let (compare Tables 2.2 and 2.3)

$$\Lambda_{[j]} = \begin{cases} \Lambda_{(j)} & \text{for } j = 1,2,\ldots,\ell-2 \ \text{ or } \ j = \ell-1,\ N = 2\ell+1\,, \\ \Lambda_{(\ell-1)} + \Lambda_{(\ell)} & \text{for } j = \ell-1,\ N = 2\ell\,, \\ 2\Lambda_{(\ell)} & \text{for } j = \ell,\ N = 2\ell+1\,, \end{cases}$$

with the fundamental weights $\Lambda_{(i)}$ as defined in Chapter 2, Subsection 2.2.1, while $\Lambda_\circ = 0$, $\Lambda_{\rm v} = 2\Lambda_{(1)}$, and, for $N = 2\ell$, $\Lambda_{\rm s} = 2\Lambda_{(\ell)}$, $\Lambda_{\rm c} = 2\Lambda_{(\ell-1)}$. We now claim

Theorem 4.2 *For $n = 0,1,2,\ldots$ the vectors of Definition 4.1 $|\Omega_\circ^{n,\pm}\rangle$, $|\Omega_{\rm v}^{n,\pm}\rangle$, $|\Omega_{[j]}^{n,\pm}\rangle$ and $|\overline{\Omega}_{[j]}^{n,\pm}\rangle$, $j = 1,2,\ldots,\ell-1$, are highest weight states of $\widehat{\mathfrak{so}}(N)_2$ with highest weights $\Lambda_\circ$, $\Lambda_{\rm v}$, $\Lambda_{[j]}$ and $\Lambda_{[j]}$, respectively; for $N = 2\ell$ the vectors $|\Omega_{\rm s}^{n,\pm}\rangle$ and $|\Omega_{\rm c}^{n,\pm}\rangle$ are highest weight states with highest weights $\Lambda_{\rm s}$ and $\Lambda_{\rm c}$, respectively, and for $N = 2\ell+1$ the vectors $|\Omega_{[\ell]}^{n,\pm}\rangle$ and $|\overline{\Omega}_{[\ell]}^{n,\pm}\rangle$ are highest weight states with highest weights $\Lambda_{[\ell]}$.*

Proof. Firstly, we have to show that all these vectors are annihilated by $\mathcal{E}_+^j$ for $j = 0,1,\ldots,\ell$. This can easily be checked by inserting the results (4.12)–(4.14) for the commutators between the step operators $\mathcal{E}_+^j$ and the operators

$X_r^{k,\pm}$, $\bar{X}_r^{k,\pm}$ into the definitions of these states. The least trivial case occurs for $\mathcal{E}_+^0$, where one employs the first of the identities (4.14); one then has to commute $\bar{x}_{1/2}^{1,\pm}$ and $\bar{x}_{1/2}^{2,\pm}$, to the right and use $\bar{x}_{1/2}^{1,\pm}|\Omega_\circ\rangle = 0 = \bar{x}_{1/2}^{2,\pm}|\Omega_\circ\rangle$ when $n = 0$, while for $n > 0$ one also must employ the second identity in (4.14). Secondly, we have to show that the states defined in Definition 4.1 are eigenvectors of all Cartan subalgebra generators $\mathcal{H}^k$ ($k = 1, 2, \dots, \ell$). More precisely, from the commutation relations (4.10) and (4.11) it follows rather directly that

$$\begin{aligned}\mathcal{H}^k\,|\Omega_{[j]}^{n,\pm}\rangle &= (\Lambda_{[j]})^k\,|\Omega_{[j]}^{n,\pm}\rangle \quad \text{for } j = 1, 2, \dots, \ell\,,\\ \mathcal{H}^k\,|\overline{\Omega}_{[j]}^{n,\pm}\rangle &= (\Lambda_{[j]})^k\,|\overline{\Omega}_{[j]}^{n,\pm}\rangle \quad \text{for } j = 1, 2, \dots, \ell-1\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}^k\,|\Omega_\circ^{n,\pm}\rangle &= (\Lambda_\circ)^k\,|\Omega_\circ^{n,\pm}\rangle\,, \qquad & \mathcal{H}^k\,|\Omega_{\mathrm{v}}^{n,\pm}\rangle &= (\Lambda_{\mathrm{v}})^k\,|\Omega_{\mathrm{v}}^{n,\pm}\rangle\,,\\ \mathcal{H}^k\,|\Omega_{\mathrm{s}}^{n,\pm}\rangle &= (\Lambda_{\mathrm{s}})^k\,|\Omega_{\mathrm{s}}^{n,\pm}\rangle\,, \qquad & \mathcal{H}^k\,|\Omega_{\mathrm{c}}^{n,\pm}\rangle &= (\Lambda_{\mathrm{c}})^k\,|\Omega_{\mathrm{c}}^{n,\pm}\rangle\,,\end{aligned}$$

the theorem is proven. □

We further claim

Theorem 4.3 *For $n = 0, 1, 2, \dots$ the vectors of Definition 4.1 $|\Omega_\circ^{n,\pm}\rangle$, $|\Omega_{\mathrm{v}}^{n,\pm}\rangle$ and $|\Omega_{[j]}^{n,\pm}\rangle$, $|\overline{\Omega}_{[j]}^{n,\pm}\rangle$, $j = 1, 2, \dots, \ell-1$, are highest weight states of the coset Virasoro algebra $\mathfrak{Vir}^{\mathrm{c}}$ with coset conformal weights*

$$\Delta_{n;\circ}^{\mathrm{c}} = \Delta_{n;\mathrm{v}}^{\mathrm{c}} = \frac{n^2 N}{2}\,, \tag{4.21}$$

and

$$\begin{aligned}\Delta_{n;j}^{\mathrm{c}} &= \frac{1}{2N}(nN + j)^2\,, & j &= 1, 2, \dots, \ell-1\,,\\ \bar{\Delta}_{n;j}^{\mathrm{c}} &= \frac{1}{2N}((n+1)N - j)^2\,, & j &= 1, 2, \dots, \ell-1\,,\end{aligned} \tag{4.22}$$

respectively; for $N = 2\ell$ the vectors $|\Omega_{\mathrm{s}}^{n,\pm}\rangle$ and $|\Omega_{\mathrm{c}}^{n,\pm}\rangle$ are highest weight states with coset conformal weights

$$\Delta_{n;\mathrm{s}}^{\mathrm{c}} = \Delta_{n;\mathrm{c}}^{\mathrm{c}} = \frac{1}{2N}(nN + \ell)^2\,, \tag{4.23}$$

respectively, and for $N = 2\ell + 1$ the vectors $|\Omega_{[\ell]}^{n,\pm}\rangle$, $|\overline{\Omega}_{[\ell]}^{n,\pm}\rangle$ are highest weight states with coset conformal weights

$$\Delta_{n;\ell}^{\mathrm{c}} = \frac{1}{2N}(nN + \ell)^2\,, \qquad \bar{\Delta}_{n;\ell}^{\mathrm{c}} = \frac{1}{2N}((n+1)N - \ell)^2\,, \tag{4.24}$$

respectively.

Proof. As a consequence of Theorem 4.2 the vectors (4.1) are highest weight states of the Sugawara Virasoro algebra. Hence we have to show that L_m^{NS} with $m > 0$ annihilates these states, which is a consequence of

$$[L_m^{\text{NS}}, x_r^{j,\pm}] = -(r + \tfrac{m}{2})\, x_{r+m}^{j,\pm}\,, \qquad [L_m^{\text{NS}}, \bar{x}_r^{j,\pm}] = -(r + \tfrac{m}{2})\, \bar{x}_{r+m}^{j,\pm}\,.$$

In particular we have

$$[L_0^{\text{NS}}, x_r^{i,\pm}] = -r\, x_r^{i,\pm}\,, \qquad [L_0^{\text{NS}}, \bar{x}_r^{i,\pm}] = -r\, \bar{x}_r^{i,\pm}\,. \tag{4.25}$$

From these relations we also deduce that

$$L_0^{\text{NS}}\, |\Omega_{[j]}^{n,\pm}\rangle = \Delta_{n;j}^{\text{NS}}\, |\Omega_{[j]}^{n,\pm}\rangle\,,$$

with conformal weights

$$\Delta_{n;j}^{\text{NS}} = [\tfrac{1}{2} + \tfrac{3}{2} + \ldots + \big(n - \tfrac{1}{2}\big)]N + \big(n + \tfrac{1}{2}\big)j = \frac{n^2 N}{2} + \big(n + \tfrac{1}{2}\big)j$$

for $j = 1, 2, \ldots, \ell$. Similarly,

$$L_0^{\text{NS}}\, |\overline{\Omega}_{[j]}^{n,\pm}\rangle = \bar{\Delta}_{n;j}^{\text{NS}}\, |\overline{\Omega}_{[j]}^{n,\pm}\rangle\,, \qquad \bar{\Delta}_{n;j}^{\text{NS}} = \frac{(n{+}1)^2 N}{2} - \big(n + \tfrac{1}{2}\big)j\,,$$

for $j = 1, 2, \ldots, \ell$. Also, for the sectors labelled by $\circ$, v, s and c we find

$$\Delta_{n;\circ}^{\text{NS}} = \frac{n^2 N}{2}\,, \qquad \Delta_{n;\text{v}}^{\text{NS}} = \frac{n^2 N}{2} + 1\,, \qquad \Delta_{n;\text{s}}^{\text{NS}} = \Delta_{n;\text{c}}^{\text{NS}} = \Delta_{n;\ell}^{\text{NS}}\,.$$

Furthermore, the conformal weights of the vectors (4.15) – (4.20) with respect to the Virasoro algebra of the level 2 WZW theory follow immediately from the $\mathfrak{so}(N)$-weights Λ by the Sugawara formula for the Virasoro generator L_0. This yields the conformal weights that were already listed in the Tables 2.2 and 2.3. By comparison of these conformal dimensions with the ones obtained above, the proof is completed. □

Since the affine Lie algebra $\widehat{\mathfrak{so}}(N)_2$ and the coset Virasoro algebra commute, it follows immediately that further highest weight vectors of $\widehat{\mathfrak{so}}(N)_2$ are obtained when acting with the lowering operators of the coset Virasoro algebra on the vectors (4.15) – (4.20). For example, applying the coset Virasoro operator L_{-1}^{c} to the highest weight vector $|\Omega_{[1]}^{+}\rangle$ we get the highest weight vector (computed for the case $N = 2\ell$)

$$L_{-1}^{\text{c}}|\Omega_{[1]}^{+}\rangle = \frac{1}{N}\left[x_{-3/2}^{1,+}|\Omega_\circ\rangle + \sum_{k=1}^{\ell}(\bar{x}_{-1/2}^{k,+}x_{-1/2}^{k,-} - \bar{x}_{-1/2}^{k,-}x_{-1/2}^{k,+})|\Omega_{[1]}^{+}\rangle\right]$$

of $\widehat{\mathfrak{so}}(N)_2$.

Also note that by construction the tensor product module, and hence each of its submodules, is unitary. Thus in particular the highest weight modules that are obtained by acting with arbitrary polynomials in the lowering operators $\mathcal{E}^j_-$ on the highest weight vectors are unitary, and hence are fully reducible.

4.2.3 $O(2)$-Transformation Properties

There is an interesting association of the $\widehat{\mathfrak{so}}(N)_2$ highest weight modules appearing in $\hat{\mathcal{H}}_{\mathrm{NS}}$ to the sectors of $\mathfrak{A}$ labelled by the spectrum of the gauge group $O(2)$. This becomes apparent from the $O(2)$-transformation properties of the highest weight vectors of Definition 4.1.

For the Fourier modes $c^{j,\pm}_r$ and $\bar{c}^{j,\pm}_r$ the action of $\varrho_{U(\gamma_t)}$, $t \in \mathbb{R}$, and of $\varrho_{U(\eta)}$ read (recall that the action of these Bogoliubov automorphisms extends to $\mathfrak{B}(\hat{\mathcal{H}}_{\mathrm{NS}})$)

$$\begin{aligned} \varrho_{U(\gamma_t)}(c^{j,\pm}_r) &= \cos(t)\, c^{j,\pm}_r - \sin(t)\, \bar{c}^{j,\pm}_r \,, \qquad & \varrho_{U(\eta)}(c^{j,\pm}_r) &= c^{j,\pm}_r \,, \\ \varrho_{U(\gamma_t)}(\bar{c}^{j,\pm}_r) &= \sin(t)\, c^{j,\pm}_r + \cos(t)\, \bar{c}^{j,\pm}_r \,, \qquad & \varrho_{U(\eta)}(\bar{c}^{j,\pm}_r) &= -\bar{c}^{j,\pm}_r \,, \end{aligned}$$

so that the combinations $x^{j,\pm}_r$ transform as

$$\varrho_{U(\gamma_t)}(x^{j,\pm}_r) = \mathrm{e}^{\pm\mathrm{i}t}\, x^{j,\pm}_r \,, \quad \varrho_{U(\eta)}(x^{j,\pm}_r) = x^{j,\mp}_r \,.$$

Analogously,

$$\varrho_{U(\gamma_t)}(\bar{x}^{j,\pm}_r) = \mathrm{e}^{\pm\mathrm{i}t}\, \bar{x}^{j,\pm}_r \,, \quad \varrho_{U(\eta)}(\bar{x}^{j,\pm}_r) = \bar{x}^{j,\mp}_r \,.$$

Hence the combinations $X^{j,\pm}_r$ transform as

$$\varrho_{U(\gamma_t)}(X^{j,\pm}_r) = \mathrm{e}^{\pm\mathrm{i}jt}\, X^{j,\pm}_r \,, \quad \varrho_{U(\eta)}(X^{j,\pm}_r) = X^{j,\mp}_r \,,$$

and analogously,

$$\varrho_{U(\gamma_t)}(\bar{X}^{j,\pm}_r) = \mathrm{e}^{\pm\mathrm{i}(\ell-j)t}\, \bar{X}^{j,\pm}_r \,, \quad \varrho_{U(\eta)}(\bar{X}^{j,\pm}_r) = \bar{X}^{j,\mp}_r \,.$$

The vacuum $|\Omega_\circ\rangle$ is $O(2)$-invariant. We then deduce the following transformations for the vectors of Definition 4.1. For all $n = 0, 1, 2, \ldots$ we have

$$Q(\gamma_t)\, |\Omega^{n,\pm}_{[j]}\rangle = \mathrm{e}^{\pm\mathrm{i}(nN+j)t}\, |\Omega^{n,\pm}_{[j]}\rangle \,, \qquad Q(\eta)\, |\Omega^{n,\pm}_{[j]}\rangle = |\Omega^{n,\mp}_{[j]}\rangle$$

for $j = 1, 2, \dots, \ell$, and

$$Q(\gamma_t)\, |\overline{\Omega}^{n,\pm}_{[j]}\rangle = \mathrm{e}^{\pm \mathrm{i}((n+1)N-j)t}\, |\overline{\Omega}^{n,\pm}_{[j]}\rangle\,, \qquad Q(\eta)\, |\overline{\Omega}^{n,\pm}_{[j]}\rangle = |\overline{\Omega}^{n,\mp}_{[j]}\rangle$$

for $j = 1, 2, \dots, \ell$. Also

$$\begin{aligned}
Q(\gamma_t)\, |\Omega^{n,\pm}_{\mathrm{o}}\rangle &= \mathrm{e}^{\pm \mathrm{i}nNt}\, |\Omega^{n,\pm}_{\mathrm{o}}\rangle\,, & Q(\eta)\, |\Omega^{n,\pm}_{\mathrm{o}}\rangle &= |\Omega^{n,\mp}_{\mathrm{o}}\rangle\,,\\
Q(\gamma_t)\, |\Omega^{n,\pm}_{\mathrm{v}}\rangle &= \mathrm{e}^{\pm \mathrm{i}nNt}\, |\Omega^{n,\pm}_{\mathrm{v}}\rangle\,, & Q(\eta)\, |\Omega^{n,\pm}_{\mathrm{v}}\rangle &= |\Omega^{n,\mp}_{\mathrm{v}}\rangle\,,\\
Q(\gamma_t)\, |\Omega^{n,\pm}_{\mathrm{s}}\rangle &= \mathrm{e}^{\pm \mathrm{i}(nN+\ell)t}\, |\Omega^{n,\pm}_{\mathrm{s}}\rangle\,, & Q(\eta)\, |\Omega^{n,\pm}_{\mathrm{s}}\rangle &= |\Omega^{n,\mp}_{\mathrm{s}}\rangle\,,\\
Q(\gamma_t)\, |\Omega^{n,\pm}_{\mathrm{c}}\rangle &= \mathrm{e}^{\pm \mathrm{i}(nN+\ell)t}\, |\Omega^{n,\pm}_{\mathrm{c}}\rangle\,, & Q(\eta)\, |\Omega^{n,\pm}_{\mathrm{c}}\rangle &= |\Omega^{n,\mp}_{\mathrm{c}}\rangle\,.
\end{aligned}$$

We remark that the highest weight states $|\Omega^{n,\pm}_{\mathrm{v}}\rangle$ and $|\Omega^{n,\pm}_{\mathrm{o}}\rangle$, $n = 1, 2, \dots$, and for even N also $|\Omega^{n,\pm}_{\mathrm{c}}\rangle$ and $|\Omega^{n,\pm}_{\mathrm{s}}\rangle$, $n = 0, 1, 2, \dots$, are connected by $O(2)$-invariant fermion bilinears, i.e. by elements of the intermediate algebra $\mathfrak{A}$. Explicitly, we have

$$|\Omega^{n,\pm}_{\mathrm{v}}\rangle = a^n_{\mathrm{v}}\, |\Omega^{n,\pm}_{\mathrm{o}}\rangle, \qquad a^n_{\mathrm{v}} = -(x^{1,-}_{n-1/2} x^{1,+}_{-n-1/2} + x^{1,+}_{n-1/2} x^{1,-}_{-n-1/2})\,,$$

for $n = 1, 2, \dots$, and

$$|\Omega^{n,\pm}_{\mathrm{c}}\rangle = a^n_{\mathrm{c}}\, |\Omega^{n,\pm}_{\mathrm{s}}\rangle, \qquad a^n_{\mathrm{c}} = -(\bar{x}^{\ell,-}_{n+1/2} \bar{x}^{\ell,+}_{-n-1/2} + \bar{x}^{\ell,+}_{n+1/2} \bar{x}^{\ell,-}_{-n-1/2})$$

for $n = 0, 1, 2, \dots$.

4.3 Characters

Owing to the inclusion $\mathfrak{A}_{\mathrm{WZW}} \subset \mathfrak{A}$, the irreducible sectors of the gauge invariant fermion algebra $\mathfrak{A}$ constitute modules of the observable algebra $\mathfrak{A}_{\mathrm{WZW}}$ of the WZW theory, which however are typically reducible. To determine the decomposition of the irreducible modules of the intermediate algebra $\mathfrak{A}$ into irreducible modules of $\mathfrak{A}_{\mathrm{WZW}}$ we analyze their characters and combine the result with the knowledge about the characters of the coset theory.

In the following calculations we directly use the argument $q = \exp(2\pi \mathrm{i}\tau)$ instead of the upper complex half plane variable τ; so it is always understood that $|q| < 1$. Moreover, we neglect the additional term $-c/24$ in the definition (1.5) which is conventionally added due to simpler transformation properties with respect to the modular group. For our purposes, this modification is not needed and would only cause several confusing prefactors. Thus we define the character $\chi(q)$ of a module simply as the trace of q^{L_0}.

4.3.1 $c = 1$ Orbifolds

Via the coset construction [32], one associates to any embedding of untwisted affine Lie algebras that is induced by an embedding of their horizontal subalgebras another conformal field theory, called the coset theory. Here the relevant embedding is that of $\widehat{\mathfrak{so}}(N)_2$ into $\widehat{\mathfrak{so}}(N)_1 \oplus \widehat{\mathfrak{so}}(N)_1$; the branching rules of this embedding are just the tensor product decompositions of $\widehat{\mathfrak{so}}(N)_1$-modules (compare Chapter 1, Subsection 1.2.3).

The Virasoro algebra of the coset theory is easily obtained as the difference of the Sugawara constructions of the Virasoro algebras of the affine Lie algebras. In contrast, the determination of the field contents of the coset theory is in general a difficult task (see e.g. [50, 28]). But in the case of our interest, the coset theory has conformal central charge $c = 1$, and the classification of (unitary) $c = 1$ conformal field theories is well known. In fact, one finds (compare e.g. [49]) that it is a so-called rational $c = 1$ orbifold theory, which can be obtained from the $c = 1$ theory of a free boson compactified on a circle by restriction to the invariants with respect to a $\mathbb{Z}_2$-symmetry. These conformal field theory models have been investigated in [14]; for our purposes we need only the following information.

The rational $c = 1$ $\mathbb{Z}_2$-orbifolds are labelled by a non-negative integer M. The theory at a given value of M has $M+7$ sectors; they are listed in the following table. Here we have again separated the fields which correspond

Table 4.1: Sectors of the $c = 1$ $\mathbb{Z}_2$-orbifolds

field	Δ	$\mathcal{D}$
$\circ$	0	1
v	1	1
s, c	$\frac{M}{4}$	1
$j \ \in \{1, 2, \dots, M-1\}$	$\frac{j^2}{4M}$	2
σ, τ	$\frac{1}{16}$	$\sqrt{M}$
σ', τ'	$\frac{9}{16}$	$\sqrt{M}$

to the (doubled) Neveu-Schwarz sector $\hat{\mathcal{H}}_{\rm NS}$ from the fields σ, τ, σ', τ' which involve the Ramond sector; the latter are known as "twist fields" of the orbifold theory.

The characters of the fields in the big Fock space are given by

$$\chi^{{\rm c};M}_{[j]}(q) = \frac{1}{\varphi(q)}\, \psi_{M,j}(q) \tag{4.26}$$

for $j = 1, 2, \dots, M$, where it is understood that

$$\chi^{{\rm c};M}_{\rm s}(q) = \chi^{{\rm c};M}_{\rm c}(q) = \tfrac{1}{2}\, \chi^{{\rm c};M}_{[M]}\,,$$

and by

$$\chi^{{\rm c};M}_{\circ}(q) = \frac{1}{2\varphi(q)} \left[\psi_{M,0}(q) + \psi_{1,0}(-q)\right], \quad \chi^{{\rm c};M}_{\rm v}(q) = \frac{1}{2\varphi(q)} \left[\psi_{M,0}(q) - \psi_{1,0}(-q)\right].$$

Here the functions $\psi_{M,j}$ are the infinite sums

$$\psi_{M,j}(q) = \sum_{m\in\mathbb{Z}} q^{(j+2mM)^2/4M}\,.$$

One has [37, p. 240]

$$\psi_{1,0}(-q) = \sum_{m\in\mathbb{Z}} (-1)^m\, q^{m^2} = \frac{(\varphi(q))^2}{\varphi(q^2)}\,. \tag{4.27}$$

It follows in particular that

$$\chi^{{\rm c};M}_{\circ}(q) - \chi^{{\rm c};M}_{\rm v}(q) = \frac{\varphi(q)}{\varphi(q^2)}\,, \tag{4.28}$$

and

$$\chi^{{\rm c};M}_{\circ}(q) + \chi^{{\rm c};M}_{\rm v}(q) = \frac{\psi_{M,0}(q)}{\varphi(q)}\,. \tag{4.29}$$

Note that the spectrum of WZW theories for even and odd N, displayed in Tables 2.1 – 2.3, is rather similar. However, to obtain the spectrum of the coset theory also the structure of the conjugacy classes of $\mathfrak{so}(N)$-modules plays an important rôle, and these are rather different for even and odd N.[1] As a consequence it depends on whether N is even or odd which $c = 1$ orbifold

[1] Also, for odd N in the Ramond sector an additional complication arises, namely a so-called fixed point resolution is required [49, 28].

one obtains as the coset theory. Namely, for $N=2\ell$ one finds $M=N/2=\ell$, while $M=2N$ for $N=2\ell+1$.

The decomposition of the products of level one characters looks as follows. For $N=2\ell$ we have

$$\begin{aligned}
[\chi^{(1)}_{\mathrm{o}}]^2 &= \chi^{\mathrm{c};\ell}_{\mathrm{o}}\,\chi^{(2)}_{\mathrm{o}} + \chi^{\mathrm{c};\ell}_{\mathrm{v}}\,\chi^{(2)}_{\mathrm{v}} + \sum_{\substack{2\le j\le\ell\\ j\ \mathrm{even}}} \chi^{\mathrm{c};\ell}_{[j]}\,\chi^{(2)}_{[j]}\,,\\
[\chi^{(1)}_{\mathrm{v}}]^2 &= \chi^{\mathrm{c};\ell}_{\mathrm{o}}\,\chi^{(2)}_{\mathrm{v}} + \chi^{\mathrm{c};\ell}_{\mathrm{v}}\,\chi^{(2)}_{\mathrm{o}} + \sum_{\substack{2\le j\le\ell\\ j\ \mathrm{even}}} \chi^{\mathrm{c};\ell}_{[j]}\,\chi^{(2)}_{[j]}\,,\\
\chi^{(1)}_{\mathrm{o}}\,\chi^{(1)}_{\mathrm{v}} &= \sum_{\substack{1\le j\le\ell\\ j\ \mathrm{odd}}} \chi^{\mathrm{c};\ell}_{[j]}\,\chi^{(2)}_{[j]}\,,
\end{aligned} \tag{4.30}$$

where it is understood that

$$\chi^{(2)}_{[\ell]}(q) \equiv \chi^{(2)}_{\mathrm{s}}(q) + \chi^{(2)}_{\mathrm{c}}(q)\,. \tag{4.31}$$

For $N=2\ell+1$, the tensor product decomposition reads instead

$$\begin{aligned}
[\chi^{(1)}_{\mathrm{o}}]^2 &= \chi^{\mathrm{c};2N}_{\mathrm{o}}\,\chi^{(2)}_{\mathrm{o}} + \chi^{\mathrm{c};2N}_{\mathrm{v}}\,\chi^{(2)}_{\mathrm{v}} + \sum_{\substack{2\le j\le\ell\\ j\ \mathrm{even}}} \chi^{\mathrm{c};2N}_{[2j]}\,\chi^{(2)}_{[j]} + \sum_{\substack{1\le j\le\ell\\ j\ \mathrm{odd}}} \chi^{\mathrm{c};2N}_{[2N-2j]}\,\chi^{(2)}_{[j]}\,,\\
[\chi^{(1)}_{\mathrm{v}}]^2 &= \chi^{\mathrm{c};2N}_{\mathrm{o}}\,\chi^{(2)}_{\mathrm{v}} + \chi^{\mathrm{c};2N}_{\mathrm{v}}\,\chi^{(2)}_{\mathrm{o}} + \sum_{\substack{2\le j\le\ell\\ j\ \mathrm{even}}} \chi^{\mathrm{c};2N}_{[2j]}\,\chi^{(2)}_{[j]} + \sum_{\substack{1\le j\le\ell\\ j\ \mathrm{odd}}} \chi^{\mathrm{c};2N}_{[2N-2j]}\,\chi^{(2)}_{[j]}\,,\\
\chi^{(1)}_{\mathrm{o}}\,\chi^{(1)}_{\mathrm{v}} &= \chi^{\mathrm{c};2N}_{[2N]}\left[\chi^{(2)}_{\mathrm{o}} + \chi^{(2)}_{\mathrm{v}}\right] + \sum_{\substack{2\le j\le\ell\\ j\ \mathrm{even}}} \chi^{\mathrm{c};2N}_{[2N-2j]}\,\chi^{(2)}_{[j]} + \sum_{\substack{1\le j\le\ell\\ j\ \mathrm{odd}}} \chi^{\mathrm{c};2N}_{[2j]}\,\chi^{(2)}_{[j]}\,.
\end{aligned} \tag{4.32}$$

It is worth noting that these formulae can be proven without too much effort, whereas in general it is a difficult task to write down such tensor product decompositions. Tools which are always available are the matching of conformal dimensions modulo integers as well as conjugacy class selection rules, which imply [50] so-called field identifications. In the present case, we can e.g. use the fact that the sum of conformal weights $\Delta^{(2)}_j = j(N-j)/2N$ and $\Delta^{\mathrm{c};M}_k = k^2/4M$ is (for generic N) a half-integer only if $k=j\sqrt{2M/N}$ or $k=(N-j)\sqrt{2M/N}$. Also, there is a conjugacy class selection rule which implies that the tensor product of modules in the Neveu-Schwarz sector yields only modules which are again in the Neveu-Schwarz sector, and the corresponding field identification tells us e.g. that the branching function $b^{\mathrm{c};M}_{\mathrm{v,v;v}}(q)$ coincides with $b^{\mathrm{c};M}_{\mathrm{o,o;o}}(q) = \chi^{\mathrm{c};M}_{\mathrm{o}}(q)$.

As it turns out, we are even in the fortunate situation that together with the known classification of unitary $c = 1$ conformal field theories, these informations already determine the tensor product decompositions almost completely. In particular, the value of M of the $c = 1$ orbifold is determined uniquely, and one can prove that there are not any further field identifications besides the ones implied by conjugacy class selection rules. Hence (4.30) and (4.32) can be viewed as a well-founded ansatz, and the remaining ambiguities can be resolved by checking various consistency relations which follow from the arguments that we will give in Subsections 4.3.3 and 4.3.4 below. Another possibility to deduce (4.30) and (4.32) is to employ the conformal embedding of $\widehat{\mathfrak{so}}(N)_2$ into $\widehat{\mathfrak{u}}(N)$ at level one [49], which corresponds to regarding the real fermions as real and imaginary parts of complex-valued fermions.

4.3.2 Characters for the Sectors of $\mathfrak{A}$

The characters of submodules of the space $\hat{\mathcal{H}}_{\rm NS}$, i.e. the trace of q^{L_0} over the modules, can be obtained as follows. Let P_0, P_J and $P^{\pm}_{[m]}$ denote the projections onto $\mathcal{H}_0, \mathcal{H}_J$ and $\mathcal{H}^{\pm}_{[m]}$ for $m \in \mathbb{N}$, respectively. Then the representation matrices $Q(\gamma_t)$ and $Q(\eta\gamma_t)$ of $O(2)$ decompose into projectors as

$$Q(\gamma_t) = P_0 + P_J + \sum_{m=1}^{\infty} \left[\mathrm{e}^{imt} P^{+}_{[m]} + \mathrm{e}^{-imt} P^{-}_{[m]} \right]$$

and

$$Q(\eta\gamma_t) = P_0 - P_J + \sum_{m=1}^{\infty} \left[\mathrm{e}^{imt} Q(\eta) P^{+}_{[m]} + \mathrm{e}^{-imt} Q(\eta) P^{-}_{[m]} \right].$$

It follows in particular that the projectors can be written as

$$P_0 = \tfrac{1}{4\pi} \int_0^{2\pi} \mathrm{d}t \left[Q(\gamma_t) + Q(\eta\gamma_t) \right], \qquad P_J = \tfrac{1}{4\pi} \int_0^{2\pi} \mathrm{d}t \left[Q(\gamma_t) - Q(\eta\gamma_t) \right],$$

$$P^{\pm}_{[m]} = \tfrac{1}{2\pi} \int_0^{2\pi} \mathrm{d}t \, \mathrm{e}^{\mp imt} \, Q(\gamma_t) \qquad \text{for } m \in \mathbb{N}.$$

For the irreducible $\mathfrak{A}$-sectors in $\hat{\mathcal{H}}_{\rm NS}$, the $O(2)$-transformation properties of the $x^{i,\pm}_r$ and $\bar{x}^{i,\pm}_r$ together with the action of $L^{\rm NS}_0$ (compare (4.25)) imply the

following. First,

$$\chi_0^{\rm NS}(q) \equiv {\rm tr}_{\hat{\mathcal{H}}_{\rm NS}} P_0\, q^{L_0^{\rm (NS)}} =$$
$$= \tfrac{1}{4\pi} \int_0^{2\pi} {\rm d}t \, \Big[\prod_{m=0}^{\infty} (1+{\rm e}^{{\rm i}t} q^{m+1/2})^N (1+{\rm e}^{-{\rm i}t} q^{m+1/2})^N + \prod_{m=0}^{\infty} (1-q^{2m+1})^N \Big] .$$

This can be rewritten as

$$\chi_0^{\rm NS}(q) = \tfrac{1}{4\pi} \int_0^{2\pi} {\rm d}t \, \Big[\frac{\xi(q;-{\rm e}^{{\rm i}t} q^{1/2})}{\varphi(q)} \Big]^N + \tfrac{1}{2} \Big[\frac{\varphi(q)}{\varphi(q^2)} \Big]^N ,$$

where φ is Euler's product function (3.5) and

$$\xi(q;z) = \prod_{n=1}^{\infty} \big((1-q^n)(1-q^n z^{-1})(1-q^{n-1} z) \big) .$$

Using also the identity [37, p. 240]

$$\xi(q;z) = \sum_{n\in\mathbb{Z}} (-1)^n \, q^{n(n-1)/2} z^n ,$$

we finally arrive at

$$\chi_0^{\rm NS}(q) = \frac{\Theta_{N,0}(q)}{2\,(\varphi(q))^N} + \frac{(\varphi(q))^N}{2\,(\varphi(q^2))^N} , \tag{4.33}$$

where we introduced the functions

$$\Theta_{N,m}(q) = \sum_{\substack{m_1,m_2,\ldots,m_N\in\mathbb{Z} \\ m_1+m_2+\ldots+m_N=m}} q^{(m_1^2+m_2^2+\ldots+m_N^2)/2} \;\; \equiv \sum_{\substack{\mathbf{m}\in\mathbb{Z}^N \\ \sum m_i=m}} q^{\mathbf{m}^2/2} \tag{4.34}$$

for $m\in\mathbb{Z}$.

Analogously, we find

$$\chi_J^{\rm NS}(q) \equiv {\rm tr}_{\hat{\mathcal{H}}_{\rm NS}} P_J\, q^{L_0^{\rm (NS)}} = \frac{\Theta_{N,0}(q)}{2\,(\varphi(q))^N} - \frac{(\varphi(q))^N}{2\,(\varphi(q^2))^N} \tag{4.35}$$

and

$$\chi_{[m]}^{\rm NS}(q) \equiv {\rm tr}_{\hat{\mathcal{H}}_{\rm NS}} P_{[m]}^{\pm}\, q^{L_0^{\rm (NS)}} = \frac{1}{2\pi} \int_0^{2\pi} {\rm d}t \, {\rm e}^{\mp{\rm i}mt} \Big[\frac{\xi(q;-{\rm e}^{{\rm i}t} q^{1/2})}{\varphi(q)} \Big]^N = \frac{\Theta_{N,m}(q)}{(\varphi(q))^N} \tag{4.36}$$

for $m \in \mathbb{N}$. (Note that the latter result does not depend on whether $P^+_{[m]}$ or $P^-_{[m]}$ is used, since $\Theta_{N,m}(q) = \Theta_{N,-m}(q)$.)

Expressing the integer m either as $m = nN + j$ or as $m = (n+1)N - j$ with $1 \le j \le \ell$, by shifting the summation indices we obtain the relation $\Theta_{N,nN+j}(q) = q^{nj+n^2N/2}\,\Theta_{N,j}(q)$. Hence we have

$$\chi^{\rm NS}_{[nN+j]}(q) = q^{nj+n^2N/2}\,\chi^{\rm NS}_{[j]}(q)\,; \tag{4.37}$$

in the same manner we obtain

$$\chi^{\rm NS}_{[(n+1)N-j]}(q) = q^{-(n+1)j+(n+1)^2N/2}\,\chi^{\rm NS}_{[j]}(q)\,. \tag{4.38}$$

For $j = 0$ we have instead

$$\chi^{\rm NS}_{[nN]}(q) = q^{n^2N/2}\,[\chi^{\rm NS}_0(q) + \chi^{\rm NS}_J(q)]$$

for all $n > 0$.

4.3.3 $\widehat{\mathfrak{so}}(N)_2$ Characters for Even N

When we use the information about the highest weight vectors with respect to the affine Lie algebra $\widehat{\mathfrak{so}}(N)$ at level 2 that we obtained above, we can derive the characters of the irreducible highest weight modules of $\widehat{\mathfrak{so}}(N)_2$ by comparing the decomposition (4.9) with the decompositions (4.30) and (4.32). We first consider the case $N = 2\ell$.

By comparison of (4.9) with (4.30) we find

$$\begin{aligned}\chi^{{\rm c};\ell}_{[j]}(q)\,\chi^{(2)}_{[j]}(q) &= \chi^{\rm NS}_{[j]}(q) + \chi^{\rm NS}_{[N-j]}(q) + \chi^{\rm NS}_{[N+j]}(q) + \chi^{\rm NS}_{[2N-j]}(q) + \dots\\ &\equiv \sum_{n=0}^{\infty}[\chi^{\rm NS}_{[nN+j]}(q) + \chi^{\rm NS}_{[(n+1)N-j]}(q)]\end{aligned}$$

for even j. Using (4.37) and (4.38), this becomes

$$\begin{aligned}\chi^{{\rm c};\ell}_{[j]}(q)\,\chi^{(2)}_{[j]}(q) &= \chi^{\rm NS}_{[j]}(q)\sum_{n\in\mathbb{Z}} q^{nj+n^2N/2}\\ &= q^{-j^2/2N}\,\psi_{\ell,j}(q)\,\chi^{\rm NS}_{[j]}(q) = q^{-j^2/2N}\,\psi_{\ell,j}(q)\,\frac{\Theta_{N,j}(q)}{(\varphi(q))^N}\,.\end{aligned}$$

Analogously, with (4.30) we obtain the same result for odd j. By inserting the coset characters $\chi^{{\rm c};\ell}_{[j]}$ (4.26) we then get

$$\chi^{(2)}_{[j]}(q) = q^{-j^2/2N}\,\frac{\Theta_{N,j}(q)}{(\varphi(q))^{N-1}}\,. \tag{4.39}$$

For $j = \ell$ one has to read this result with (4.31), which means that our result only describes the sum of the irreducible characters $\chi^{(2)}_{\rm s}$ and $\chi^{(2)}_{\rm c}$. By comparison with (4.36), we may also rewrite the result in the form

$$\begin{aligned} \chi^{\rm NS}_{[nN+j]}(q) &= \frac{q^{(nN+j)^2/2N}}{\varphi(q)}\,\chi^{(2)}_{[j]}(q)\,, \\ \chi^{\rm NS}_{[(n+1)N-j]}(q) &= \frac{q^{((n+1)N-j)^2/2N}}{\varphi(q)}\,\chi^{(2)}_{[j]}(q) \end{aligned} \tag{4.40}$$

for $j = 1, 2, \ldots, \ell$.

Comparing (4.9) again with (4.30), we also find

$$\begin{aligned} &\chi^{{\rm c};\ell}_{\circ}(q)\,\chi^{(2)}_{\circ}(q) + \chi^{{\rm c};\ell}_{\rm v}(q)\,\chi^{(2)}_{\rm v}(q) = \\ &\quad = \chi^{\rm NS}_0(q) + \sum_{n=1}^{\infty}\chi^{\rm NS}_{[nN]}(q) \\ &\quad = \left[\chi^{\rm NS}_0(q) + \chi^{\rm NS}_J(q)\right]\left[\tfrac{1}{2} + \tfrac{1}{2}\,\psi_{\ell,0}(q)\right] - \chi^{\rm NS}_J(q) \\ &\quad = \tfrac{1}{2}\left[\chi^{\rm NS}_0(q) - \chi^{\rm NS}_J(q)\right] + \tfrac{1}{2}\,\psi_{\ell,0}(q)\left[\chi^{\rm NS}_0(q) + \chi^{\rm NS}_J(q)\right] \\ &\quad = \frac{(\varphi(q))^N}{2\,(\varphi(q^2))^N} + \psi_{\ell,0}(q)\,\frac{\Theta_{N,0}(q)}{2\,(\varphi(q))^N} \end{aligned} \tag{4.41}$$

and

$$\begin{aligned} \chi^{{\rm c};\ell}_{\circ}(q)\,\chi^{(2)}_{\rm v}(q) + \chi^{{\rm c};\ell}_{\rm v}(q)\,\chi^{(2)}_{\circ}(q) &= \chi^{\rm NS}_J(q) + \sum_{n=1}^{\infty}\chi^{\rm NS}_{[nN]}(q) \\ &= -\frac{(\varphi(q))^N}{2\,(\varphi(q^2))^N} + \psi_{\ell,0}(q)\,\frac{\Theta_{N,0}(q)}{2\,(\varphi(q))^N}\,. \end{aligned} \tag{4.42}$$

Subtraction of (4.42) from (4.41) yields

$$\left[\chi^{{\rm c};\ell}_{\circ}(q) - \chi^{{\rm c};\ell}_{\rm v}(q)\right]\cdot\left[\chi^{(2)}_{\circ}(q) - \chi^{(2)}_{\rm v}(q)\right] = \left[\frac{\varphi(q)}{\varphi(q^2)}\right]^N \equiv \chi^{\rm NS}_0(q) - \chi^{\rm NS}_J(q)\,,$$

so that by inserting (4.28) we obtain

$$\chi^{(2)}_{\circ}(q) - \chi^{(2)}_{\rm v}(q) = \left[\frac{\varphi(q)}{\varphi(q^2)}\right]^{N-1}. \tag{4.43}$$

Analogously, by adding (4.41) and (4.42) we get

$$\left[\chi^{{\rm c};\ell}_{\circ}(q) + \chi^{{\rm c};\ell}_{\rm v}(q)\right]\cdot\left[\chi^{(2)}_{\circ}(q) + \chi^{(2)}_{\rm v}(q)\right] = \psi_{\ell,0}(q)\,\frac{\Theta_{N,0}(q)}{(\varphi(q))^N}\,,$$

and hence inserting (4.29) we obtain

$$\chi_{\circ}^{(2)}(q) + \chi_{\mathrm{v}}^{(2)}(q) = \frac{\Theta_{N,0}(q)}{(\varphi(q))^{N-1}}\,. \tag{4.44}$$

In summary, we have derived that

$$\begin{aligned} \chi_{\circ}^{(2)}(q) &= \tfrac{1}{2}\left\{\tfrac{\Theta_{N,0}(q)}{(\varphi(q))^{N-1}} + \left[\tfrac{\varphi(q)}{\varphi(q^2)}\right]^{N-1}\right\} \\ &\equiv \tfrac{1}{2\,(\varphi(q))^{N-1}}\left[\Theta_{N,0}(q) + (\psi_{1,0}(-q))^{N-1}\right], \\ \chi_{\mathrm{v}}^{(2)}(q) &= \tfrac{1}{2}\left\{\tfrac{\Theta_{N,0}(q)}{(\varphi(q))^{N-1}} - \left[\tfrac{\varphi(q)}{\varphi(q^2)}\right]^{N-1}\right\} \\ &\equiv \tfrac{1}{2\,(\varphi(q))^{N-1}}\left[\Theta_{N,0}(q) - (\psi_{1,0}(-q))^{N-1}\right]. \end{aligned} \tag{4.45}$$

Further, comparison with (4.33) and (4.35) yields

$$\chi_0^{\mathrm{NS}}(q) + \chi_J^{\mathrm{NS}}(q) = \frac{1}{\varphi(q)}\left[\chi_{\circ}^{(2)}(q) + \chi_{\mathrm{v}}^{(2)}(q)\right], \tag{4.46}$$

while comparison with (4.36) and (4.37) shows that

$$\chi_{[nN]}^{\mathrm{NS}}(q) = \frac{q^{n^2N/2}}{\varphi(q)}\left[\chi_{\circ}^{(2)}(q) + \chi_{\mathrm{v}}^{(2)}(q)\right]. \tag{4.47}$$

4.3.4 $\widehat{\mathfrak{so}}(N)_2$ Characters for Odd N

Now we consider the case $N = 2\ell + 1$. From (4.9) and (4.32) we find

$$\begin{aligned} \chi_{[2j]}^{\mathrm{c};2N}(q)\,\chi_{[j]}^{(2)}(q) &= \chi_{[j]}^{\mathrm{NS}}(q) + \chi_{[2N-j]}^{\mathrm{NS}}(q) + \chi_{[2N+j]}^{\mathrm{NS}}(q) + \chi_{[4N-j]}^{\mathrm{NS}}(q) + \ldots \\ &\equiv \sum_{n=0}^{\infty}[\chi_{[2nN+j]}^{\mathrm{NS}}(q) + \chi_{[2(n+1)N-j]}^{\mathrm{NS}}(q)] \\ &= \chi_{[j]}^{\mathrm{NS}}(q)\sum_{n\in\mathbb{Z}} q^{2nj+2n^2N} = q^{-j^2/2N}\,\psi_{2N,2j}(q)\,\chi_{[j]}^{\mathrm{NS}}(q) \end{aligned}$$

for j even, and

$$\begin{aligned}
\chi^{c;2N}_{[2N-2j]}(q)\,\chi^{(2)}_{[j]}(q) &= \sum_{n=0}^{\infty}[\chi^{\rm NS}_{[(2n+1)N+j]}(q)+\chi^{\rm NS}_{[(2n+1)N-j]}(q)] \\
&= \chi^{\rm NS}_{[j]}(q)\sum_{n\in\mathbb{Z}} q^{-(2n+1)j+(2n+1)^2N/2} \\
&= q^{-j+N/2}\chi^{\rm NS}_{[j]}(q)\sum_{n\in\mathbb{Z}} q^{2n(N-j)+2n^2N} \\
&= q^{-j^2/2N}\,\psi_{2N,2N-2j}(q)\,\chi^{\rm NS}_{[j]}(q)
\end{aligned}$$

for j odd. By inserting the coset characters (4.26) we then arrive once again at the formulae (4.39) and (4.40) for $j=1,2,\dots,\ell$.

In the same manner we find

$$\begin{aligned}
&\chi^{c;2N}_{\circ}(q)\,\chi^{(2)}_{\circ}(q)+\chi^{c;2N}_{\rm v}(q)\,\chi^{(2)}_{\rm v}(q) = \\
&\quad = \chi^{\rm NS}_{0}(q)+\sum_{n=1}^{\infty}\chi^{\rm NS}_{[2nN]}(q) \\
&\quad = [\chi^{\rm NS}_{0}(q)+\chi^{\rm NS}_{J}(q)]\left[\tfrac{1}{2}+\tfrac{1}{2}\sum_{n\in\mathbb{Z}} q^{2n^2N}\right]-\chi^{\rm NS}_{J}(q) \\
&\quad = \frac{(\varphi(q))^N}{2\,(\varphi(q^2))^N}+\psi_{2N,0}(q)\,\frac{\Theta_{N,0}(q)}{2\,(\varphi(q))^N}
\end{aligned}$$

and

$$\begin{aligned}
\chi^{c;2N}_{\circ}(q)\,\chi^{(2)}_{\rm v}(q)+\chi^{c;2N}_{\rm v}(q)\,\chi^{(2)}_{\circ}(q) &= \chi^{\rm NS}_{J}(q)+\sum_{n=1}^{\infty}\chi^{\rm NS}_{[2nN]}(q) \\
&= -\frac{(\varphi(q))^N}{2\,(\varphi(q^2))^N}+\psi_{2N,0}(q)\,\frac{\Theta_{N,0}(q)}{2\,(\varphi(q))^N}\,.
\end{aligned}$$

Thus we also obtain again the relations (4.43) and (4.44) for $\chi^{(2)}_{\circ}$ and $\chi^{(2)}_{\rm v}$, and hence also (4.45) and (4.47).

4.4 Decomposition of the Tensor Product

The comparison of the $\mathfrak{so}(N)_2$ characters with characters of the $\mathfrak{A}$-sectors will now yield the complete decomposition of the big Fock space into tensor products of irreducible $\widehat{\mathfrak{so}}(N)_2$-modules and $\mathfrak{Vir}^{\rm c}$-modules.

4.4.1 Evaluation of the Character Formulae

Let us now summarize some of our results on the tensor product decompositions. To this end we first note that $q^{\Delta}/\varphi(q)$ is precisely the character of the Verma module $M(c,\Delta)$ of the Virasoro algebra. For central charge $c=1$ the Verma module $M(c,\Delta)$ is irreducible as long as $4\Delta \neq m^2$ for $m\in\mathbb{Z}$; otherwise there exist null states. The characters of the irreducible modules $V(1,\Delta)$ of the $c=1$ Virasoro algebra are then given by

$$\chi^{\text{Vir}}_{\Delta}(q) = \begin{cases} (\varphi(q))^{-1}\,[q^{m^2/4} - q^{(m+2)^2/4}] & \text{if } \Delta = \frac{m^2}{4} \text{ with } m\in\mathbb{Z}\,, \\ (\varphi(q))^{-1}\,q^{\Delta} & \text{otherwise.} \end{cases}$$

Thus for $\Delta = m^2/4$ with $m\in\mathbb{Z}$ the Verma module character can be decomposed as follows:

$$\frac{q^{m^2/4}}{\varphi(q)} = \frac{1}{\varphi(q)}\sum_{k=0}^{\infty}\left[q^{(m+2k)^2/4} - q^{(m+2k+2)^2/4}\right] = \sum_{k=0}^{\infty}\chi^{\text{Vir}}_{(m+2k)^2/4}(q)\,.$$

Correspondingly we write

$$W(1,\Delta) = \begin{cases} \displaystyle\bigoplus_{k=0}^{\infty} V(1,\tfrac{(m+2k)^2}{4}) & \text{if } \Delta = \frac{m^2}{4} \text{ with } m\in\mathbb{Z}\,, \\ V(1,\Delta) & \text{otherwise.} \end{cases} \tag{4.48}$$

Using also the formulae (4.21) and (4.24) for the coset conformal weights, we can summarize our results of Section 4.3 by the following description of the big Fock space $\hat{\mathcal{H}}_{\text{NS}}$. Recalling the decomposition

$$\hat{\mathcal{H}}_{\text{NS}} = \mathcal{H}_0 \oplus \mathcal{H}_J \oplus \bigoplus_{m=1}^{\infty}(\mathcal{H}_{[m]}\otimes\mathbb{C}^2)$$

of $\hat{\mathcal{H}}_{\text{NS}}$ into $\mathfrak{A}$-sectors, we can express the splitting of $\hat{\mathcal{H}}_{\text{NS}}$ into tensor products of the Virasoro modules (4.48) and the irreducible highest weight modules of $\widehat{\mathfrak{so}}(N)_2$ (that is, $\mathcal{H}^{(2)}_{\circ}$, $\mathcal{H}^{(2)}_{\text{v}}$, $\mathcal{H}^{(2)}_{[j]}$, and also $\mathcal{H}^{(2)}_{\text{s}}$ and $\mathcal{H}^{(2)}_{\text{c}}$ when $N=2\ell$) as follows. Our results show

Theorem 4.4 *For the $\mathfrak{A}$-sectors $\mathcal{H}_{[m]}$, $m=1,2,\ldots$, we have*

$$\mathcal{H}_{[nN]} = [\mathcal{H}^{(2)}_{\circ}\oplus\mathcal{H}^{(2)}_{\text{v}}]\otimes W(1,\Delta^{\text{c}}_{n;\circ})\,, \tag{4.49}$$

for $n = 1, 2, \ldots$, as well as

$$\begin{aligned} \mathcal{H}_{[nN+j]} &= \mathcal{H}^{(2)}_{[j]} \otimes W(1, \Delta^{\mathrm{c}}_{n;j}) \,, \\ \mathcal{H}_{[(n+1)N-j]} &= \mathcal{H}^{(2)}_{[j]} \otimes W(1, \bar{\Delta}^{\mathrm{c}}_{n;j}) \end{aligned} \tag{4.50}$$

for $n = 0, 1, \ldots$ and $j = 1, 2, \ldots, \ell - 1$. When $N = 2\ell + 1$, (4.50) also holds for $j = \ell$, while for $j = \ell$ and $N = 2\ell$ we have

$$\mathcal{H}_{[nN+\ell]} = [\mathcal{H}^{(2)}_{\mathrm{s}} \oplus \mathcal{H}^{(2)}_{\mathrm{c}}] \otimes W(1, \Delta^{\mathrm{c}}_{n;\mathrm{s}}) \tag{4.51}$$

for $n = 0, 1, \ldots$. The modules $W(1, \Delta)$ appearing in these decompositions are all irreducible as long as $\sqrt{2N} \notin \mathbb{N}$. Otherwise we can write $N = 2K^2$ with $K \in \mathbb{N}$, and then the modules $W(1, \Delta^{\mathrm{c}}_{n;\circ})$ and $W(1, \Delta^{\mathrm{c}}_{n;j})$, $W(1, \bar{\Delta}^{\mathrm{c}}_{n;j})$ with $j = mK$, $m = 1, 2, \ldots$ and $j \leq \ell$, split up as in (4.48).

Besides the coset Virasoro generators, the chiral symmetry algebra of the orbifold coset theory contains further operators [14]. The observation above implies in particular that when acting on $\mathfrak{A}$-sectors other than $\mathcal{H}_0$ and $\mathcal{H}_J$, for $\sqrt{2N} \notin \mathbb{N}$ all these additional generators make transitions between the sectors of the gauge invariant fermion algebra $\mathfrak{A}$; for $N = 2K^2$ ($K \in \mathbb{N}$) the additional generators generically still make transitions, except that they can map sectors with $j = mK$ to themselves. It follows in particular that we can distinguish between elements of the coset Virasoro algebra and elements of the full coset chiral algebra which are not contained in the coset Virasoro algebra by acting with them on suitable $\mathfrak{A}$-sectors.

4.4.2 The Sectors $\mathcal{H}_0$ and $\mathcal{H}_J$

It still remains to analyze the decomposition of the $\mathfrak{A}$-sectors $\mathcal{H}_0$ and $\mathcal{H}_J$ explicitly. From (4.46) we conclude that

$$\mathcal{H}_0 \oplus \mathcal{H}_J = \left[\mathcal{H}^{(2)}_{\circ} \oplus \mathcal{H}^{(2)}_{\mathrm{v}}\right] \otimes W(1, 0) \,. \tag{4.52}$$

Now $W(1, 0)$ is always reducible, independent of the particular value of the integer N. We first claim

Lemma 4.5 *The characters* χ_0^{NS} *and* χ_J^{NS} *decompose as follows:*

$$\begin{aligned} \chi_0^{\text{NS}} &= \chi_\circ^{(2)} \cdot \sum_{k=0}^{\infty} \chi^{\text{Vir}}_{(2k)^2} + \chi_{\text{v}}^{(2)} \cdot \sum_{k=0}^{\infty} \chi^{\text{Vir}}_{(2k+1)^2} \,, \\ \chi_J^{\text{NS}} &= \chi_\circ^{(2)} \cdot \sum_{k=0}^{\infty} \chi^{\text{Vir}}_{(2k+1)^2} + \chi_{\text{v}}^{(2)} \cdot \sum_{k=0}^{\infty} \chi^{\text{Vir}}_{(2k)^2} \,. \end{aligned} \tag{4.53}$$

Proof. We compute

$$\begin{aligned} \chi_0^{\text{NS}}(q) &= \frac{\Theta_{N,0}(q)}{2(\varphi(q))^N} + \frac{(\varphi(q))^{N-2}}{2(\varphi(q^2))^{N-1}} \sum_{k\in\mathbb{Z}} (-1)^k q^{k^2} \\ &\equiv \frac{\Theta_{N,0}(q)}{2(\varphi(q))^N} + \frac{(\varphi(q))^{N-2}}{2(\varphi(q^2))^{N-1}} \sum_{k=0}^{\infty} \left[q^{(2k)^2} - 2q^{(2k+1)^2} + q^{(2k+2)^2}\right] \\ &= \chi_\circ^{(2)}(q) \frac{1}{\varphi(q)} \sum_{k=0}^{\infty} \left[q^{(2k)^2} - q^{(2k+1)^2}\right] + \\ &\qquad\qquad + \chi_{\text{v}}^{(2)}(q) \frac{1}{\varphi(q)} \sum_{k=0}^{\infty} \left[q^{(2k+1)^2} - q^{(2k+2)^2}\right] \\ &\equiv \chi_\circ^{(2)}(q) \cdot \sum_{k=0}^{\infty} \chi^{\text{Vir}}_{(2k)^2}(q) + \chi_{\text{v}}^{(2)}(q) \cdot \sum_{k=0}^{\infty} \chi^{\text{Vir}}_{(2k+1)^2}(q) \end{aligned}$$

(in the first line we used (4.27)), and analogously for χ_J^{NS}. □

Hence we arrive at

Theorem 4.6 *For the* $\mathfrak{A}$*-sectors* $\mathcal{H}_0$ *and* $\mathcal{H}_J$ *we have*

$$\begin{aligned} \mathcal{H}_0 &= \mathcal{H}_\circ^{(2)} \otimes \bigoplus_{k=0}^{\infty} V(1,(2k)^2) \;\; \oplus \;\; \mathcal{H}_{\text{v}}^{(2)} \otimes \bigoplus_{k=0}^{\infty} V(1,(2k+1)^2) \,, \\ \mathcal{H}_J &= \mathcal{H}_\circ^{(2)} \otimes \bigoplus_{k=0}^{\infty} V(1,(2k+1)^2) \;\; \oplus \;\; \mathcal{H}_{\text{v}}^{(2)} \otimes \bigoplus_{k=0}^{\infty} V(1,(2k)^2) \,. \end{aligned} \tag{4.54}$$

It follows that besides $|\Omega_\circ^{0,0}\rangle \equiv |\Omega_\circ\rangle$ and $|\Omega_{\text{v}}^{J,0}\rangle \equiv |\Omega_{\text{v}}\rangle$, there must exist further simultaneous highest weight vectors of $\widehat{\mathfrak{so}}(N)_2$ and the coset Virasoro algebra, namely, for $k = 0, 1, 2, \ldots$, highest weight vectors $|\Omega_\circ^{0,2k+2}\rangle$, $|\Omega_{\text{v}}^{0,2k+1}\rangle \in$

$\mathcal{H}_0$ and $|\Omega_\circ^{J,2k+1}\rangle$, $|\Omega_{\rm v}^{J,2k+2}\rangle \in \mathcal{H}_J$, with $\widehat{\mathfrak{so}}(N)_2$-weights $\Lambda_\circ$, $\Lambda_{\rm v}$, $\Lambda_\circ$, $\Lambda_{\rm v}$, respectively, and with coset conformal weights $(2k+2)^2$, $(2k+1)^2$, $(2k+1)^2$, $(2k+2)^2$, respectively. Those vectors with unit coset conformal weight have a relatively simple form. Define

$$Z_{-1/2} = \begin{cases} \displaystyle\sum_{k=1}^{\ell} \left(\bar{x}^{k,+}_{-1/2} x^{k,-}_{-1/2} - \bar{x}^{k,-}_{-1/2} x^{k,+}_{-1/2}\right) & \text{for } N = 2\ell\,, \\ \displaystyle\sum_{k=1}^{\ell} \left(\bar{x}^{k,+}_{-1/2} x^{k,-}_{-1/2} - \bar{x}^{k,-}_{-1/2} x^{k,+}_{-1/2}\right) + \bar{x}^{\ell+1,+}_{-1/2}\, \bar{x}^{\ell+1,-}_{-1/2} & \text{for } N = 2\ell+1\,. \end{cases}$$

Then

$$|\Omega_\circ^{J,1}\rangle = Z_{-1/2}\, |\Omega_\circ\rangle$$

as well as

$$|\Omega_{\rm v}^{0,1}\rangle = \left[x^{1,+}_{-1/2} x^{1,-}_{-1/2} Z_{-1/2} + x^{1,+}_{-3/2} x^{1,-}_{-1/2} + x^{1,-}_{-3/2} x^{1,+}_{-1/2}\right] |\Omega_\circ\rangle\,.$$

In contrast, the highest weight vectors with larger coset conformal weight are more difficult to identify.

4.4.3 A Comparison of Algebras

From gauge invariance and also from the decomposition of $\hat{\mathcal{H}}_{\rm NS}$ one can deduce a lot of information about inclusions of the algebras of bounded operators that are associated to several Lie algebras acting in the big Fock space. By $\mathfrak{A}_{\mathfrak{Cos}}$ and $\mathfrak{A}_{\mathfrak{Vir}^{\rm c}}$ we denote the C^*-algebras associated to the full coset chiral algebra $\mathfrak{Cos} = (\widehat{\mathfrak{so}}(N)_1 \oplus \widehat{\mathfrak{so}}(N)_1)/\widehat{\mathfrak{so}}(N)_2$ and its Virasoro subalgebra $\mathfrak{Vir}^{\rm c}$, respectively. Clearly we have $\mathfrak{A}_{\mathfrak{Cos}} \subset \mathfrak{A}'_{\rm WZW}$. It follows by gauge invariance of $\mathfrak{Vir}^{\rm c}$ that

$$\mathfrak{A}_{\mathfrak{Vir}^{\rm c}} \subset \mathfrak{A}\,.$$

Since the full coset algebra $\mathfrak{Cos}$ involves different $\mathfrak{A}$-sectors we have in contrast

$$\mathfrak{A}_{\mathfrak{Cos}} \not\subset \mathfrak{A}\,,$$

i.e. not the whole operator content of the coset chiral algebra is $O(2)$ invariant. Furthermore, we have by gauge invariance

$$\mathfrak{A}_{\rm WZW} \cup \mathfrak{A}_{\mathfrak{Vir}^{\rm c}} \subset \mathfrak{A}\,.$$

However, this is a proper inclusion since at least the multiplicity space $W(1,0)$ in (4.52) is reducible. In other words, there is a gauge invariant operator content in $\mathfrak{A}_{\mathfrak{Cos}}$ besides $\mathfrak{A}_{\mathfrak{Vir}^c}$ which may be enlarged if $\sqrt{2N} \in \mathbb{Z}$.

4.5 Remarks

We conclude this chapter with some general remarks on our analysis.

4.5.1 Remarks on the Characters

Our idea to employ the representation theory of the gauge group $O(2)$ allowed us to deduce simple formulae for the characters of the (Neveu-Schwarz sector) irreducible highest weight modules of $\widehat{\mathfrak{so}}(N)$ at level 2. They are given by the expressions (4.39) for $\chi^{(2)}_{[j]}$ and (4.45) for $\chi^{(2)}_{\circ}$ and $\chi^{(2)}_{\mathrm{v}}$. Note that, not surprisingly, these results have a simple functional dependence on the integer N, even though the details of their derivation (involving e.g. the relation with the orbifold coset theory) depend quite non-trivially on whether N is even or odd.

Our results for these characters are not new. In [49], the conformal embedding of $\widehat{\mathfrak{so}}(N)_2$ into $\widehat{\mathfrak{u}}(N)$ at level 1 was employed to identify (sums of) $\widehat{\mathfrak{so}}(N)_2$ characters with characters of $\widehat{\mathfrak{su}}(N)_1$. Indeed, the restricted summation over the lattice vector $\mathbf{m} \in \mathbb{Z}^N$ in the formula (4.34) for $\Theta_{N,m}(q)$ precisely corresponds to the summation over the appropriately shifted root lattice of $\mathfrak{su}(N)$.

With the help of the conformal embedding only the linear combination $\chi^{(2)}_{\circ} + \chi^{(2)}_{\mathrm{v}}$ of the irreducible characters $\chi^{(2)}_{\circ}$ and $\chi^{(2)}_{\mathrm{v}}$ is obtained, which is just the level 1 vacuum character of $\widehat{\mathfrak{su}}(N)$. However, the orthogonal linear combination $\chi^{(2)}_{\circ} - \chi^{(2)}_{\mathrm{v}}$ is known as well; it has been obtained in [38, p. 233] by making use of the theory of modular forms.

4.5.2 A Homomorphism of Fusion Rings

In the previous section we were able to identify the $\widehat{\mathfrak{so}}(N)_2$ highest weight modules within the sectors of the intermediate algebra $\mathfrak{A}$ which are governed by the gauge group $O(2)$. Our results amount to the following assignment ρ

of the $O(2)$-representations to the WZW sectors:

$$
\begin{aligned}
&\rho(\Phi_0) = 1\,, \qquad \rho(\Phi_J) = \mathrm{v}\,,\\
&\rho(\Phi_{[(n+1)N]}) = 1 + \mathrm{v}\,,\\
&\rho(\Phi_{[nN+j]}) = \rho(\Phi_{[(n+1)N-j]}) = \phi_{[j]} \qquad \text{for } j = 1, 2, \dots, \ell-1\,,\\
&\rho(\Phi_{[nN+\ell]}) = \rho(\Phi_{[(n+1)N-\ell]}) = \begin{cases} \mathrm{s}+\mathrm{c} & \text{for } N = 2\ell\,,\\ \phi_{[\ell]} & \text{for } N = 2\ell+1\,, \end{cases}
\end{aligned}
$$

for $n = 0, 1, 2, \dots$ (Note that in the case of Φ_0 and Φ_J, the action of ρ does not directly correspond to the decomposition of the $\mathfrak{A}$-sectors into $\widehat{\mathfrak{so}}(N)_2$ sectors.)

The multiplication rules of the representation ring $\mathcal{R}_{O(2)}$ of $O(2)$ are given by the relations (4.7). The level 2 WZW sectors generate a fusion ring, too, which we denote by $\mathcal{R}^{(2)}_{\mathrm{WZW}}$. The ring $\mathcal{R}^{(2)}_{\mathrm{WZW}}$ has a fusion subring $\mathcal{R}^{(2)}_{\mathrm{NS}}$ which is generated by those primary fields which appear in the big Fock space $\hat{\mathcal{H}}_{\mathrm{NS}}$. The fusion rules, i.e. the structure constants of $\mathcal{R}^{(2)}_{\mathrm{WZW}}$, can be computed with the help of the Kac–Walton and Verlinde formulae (see e.g. [29]).

Inspection shows that $\mathcal{R}^{(2)}_{\mathrm{NS}}$ is in fact isomorphic to the representation ring of the dihedral group $\mathcal{D}_N$. Now for any N the group $\mathcal{D}_N$ is a finite subgroup of $O(2)$. As a consequence, the mapping ρ actually constitutes a fusion ring *homomorphism* from the representation ring $\mathcal{R}_{O(2)}$ of $O(2)$ to the fusion subring $\mathcal{R}^{(2)}_{\mathrm{NS}}$ of $\mathcal{R}^{(2)}_{\mathrm{WZW}}$. (It is also easily checked that for odd N the homomorphism ρ is surjective, while for even N the image does not contain the linear combination $\mathrm{s}-\mathrm{c}$.) This observation explains to a certain extent why, in spite of the fact that the WZW observable algebra $\mathfrak{A}_{\mathrm{WZW}}$ is much smaller than the $O(2)$-invariant algebra $\mathfrak{A}$, the group $O(2)$ nevertheless provides a substitute for the gauge group in the DHR sense. But even in view of this relationship it is still surprising how closely the WZW superselection structure follows the representation theory of $O(2)$.

One may speculate that the presence of the homomorphism ρ indicates that the gauge group $O(2)$ is in fact part of the full (as yet unknown) quantum symmetry of the WZW theory that fully takes over the rôle of the DHR gauge group. This is possible because all sectors in the Neveu-Schwarz part of the WZW theory have integral quantum dimension. This is however a rather special situation as in rational conformal field theory sectors with integral quantum dimension are actually extremely rare.

4.5.3 Discussion and Outlook

It would be desirable to incorporate also the twisted sectors $\sigma, \sigma', \tau, \tau'$ ($N = 2\ell$) respectively σ, σ' ($N = 2\ell + 1$) in our analysis. These modules appear in the tensor products that contain also the Ramond sector of the level 1 theory. In order to avoid severe technical difficulties we did not treat Ramond fermions here. More precisely, we expect the twisted sectors to be realized in the tensor product $\mathcal{H}_{\mathrm{NS}} \otimes \mathcal{H}_{\mathrm{R}}$. When one tries to incorporate this space in our analysis several unsolved problems arise. The level 2 currents acting in $\mathcal{H}_{\mathrm{NS}} \otimes \mathcal{H}_{\mathrm{R}}$ are of the form $J^{\mathrm{NS}} + J^{\mathrm{R}}$ where each summand acts non-trivially on the corresponding tensor factor. Since J^{NS} is constructed from Neveu-Schwarz fermions and J^{R} from Ramond fermions the $O(2)$-invariance is less obvious. However, there is an argument coming from the framework of bounded operators that states that $O(2)$-invariance is just hidden here: Although this is not yet proven, local normality of the local algebras of bounded operators associated to the WZW model is expected to hold also for the twisted sectors. Hence the local algebras in any sectors are isomorphic so that $O(2)$-invariance is given implicitly from the vacuum sector. Unfortunately the decomposition of $\mathcal{H}_{\mathrm{NS}} \otimes \mathcal{H}_{\mathrm{R}}$ into sectors of the gauge invariant fermion algebra cannot simply be provided as in $\hat{\mathcal{H}}_{\mathrm{NS}}$ since the associated state $\omega_{P_{\mathrm{NS}}} \otimes \omega_{S_{\mathrm{R}}}$ is neither pure nor gauge invariant. Moreover, the explicit formulae for the highest weight vectors in $\mathcal{H}_{\mathrm{R}}$ at level 1 are already much more complicated as those in $\mathcal{H}_{\mathrm{NS}}$. Therefore we believe that one needs some new ideas to treat the twisted sectors as well.

Perhaps a more hopeful task is the generalization of the analysis to higher levels $k^\vee$. Then one has to investigate the $k^\vee$-fold tensor product $\hat{\mathcal{H}}_{\mathrm{NS}} = \mathcal{H}_{\mathrm{NS}}^{\otimes k^\vee}$ which arises from the Fock representation that is associated to the basis projection $P_{\mathrm{NS}} \otimes \mathbb{1}_{k^\vee}$ of $\mathcal{K} \otimes \mathbb{C}^{k^\vee}$. The level $k^\vee$ current operators are then invariant under the gauge group $O(k^\vee)$. Owing to the more complex representation theory of the group $O(k^\vee)$ the DHR decomposition of the big Fock space $\hat{\mathcal{H}}_{\mathrm{NS}}$ into sectors of the gauge invariant fermion algebra $\mathfrak{A}$ will become more complicated. Moreover, at higher level most of the sectors have non-integral quantum dimensions. Since one cannot expect that such sectors possess a simple assignment to the $\mathfrak{A}$-sectors as it is realized in the fusion ring homomorphism at level 2, the identification of the simultaneous highest weight vectors of $\widehat{\mathfrak{so}}(N)_{k^\vee}$ and the Virasoro algebra of the coset theory $(\widehat{\mathfrak{so}}(N)_1^{\oplus k^\vee})/\widehat{\mathfrak{so}}(N)_{k^\vee}$ may be more involved. Further complications arise since the central charge of the coset Virasoro algebra then depends on N, namely

it is given by

$$c^{\mathrm{c}} = \frac{Nk^{\vee}(k^{\vee}-1)}{2(N+k^{\vee}-2)} \,.$$

The most hopeful generalization is possibly the application of our ideas to $\mathfrak{su}(N)$ WZW models. Fortunately, no Ramond fermions are needed there; all the level 1 $\widehat{\mathfrak{su}}(N)$ unitary highest weight modules are realized in one and the same Fock space even though with an infinite multiplicity. At level $k^{\vee}$, the DHR gauge group that appears is given by $U(k^{\vee})$. It will be interesting to study the relationship between the representation ring of $U(k^{\vee})$ and the WZW fusion ring in these cases where most of the sectors have non-integral quantum dimension.

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